σ-Algebra

Definition (σ-Algebra)

Let Ω be a sample space.
A collection $\mathscr{F}$ of subsets of Ω is called a *σ-algebra* if it satisfies the following properties:

1. $\Omega \in \mathscr{F}$.
2. If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$ (closed under complements).
3. If $A_1, A_2, \cdots \in \mathscr{F}$, then
$$\bigcup_{i \in \mathbb{N}} A_i \in \mathscr{F}.$$
(closed under countably infinite unions).

- Elements of a σ-algebra are called *events*.
- The pair $(\Omega, \mathscr{F})$ is called a *measurable space*.

Probability Measure

Definition (Probability Measure)

A function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is called a *probability measure* if the following properties are satisfied:

1. $\mathbb{P}(\emptyset) = 0.$
2. $\mathbb{P}(\Omega) = 1.$
3. If $A_1, A_2, \ldots$ is a countable collection of disjoint sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

Measurable Function

Definition (Measurable Function)

Let $(E, \mathscr{E})$ and $(G, \mathscr{G})$ be two measurable spaces.
Consider a function $f : E \to G$. The function f is said to be measurable if

$$\forall B \in \mathscr{G}, \qquad f^{-1}(B) = \{e \in E : f(e) \in B\} \in \mathscr{E}.$$

$$\begin{tikzcd}
(E, \mathscr{E}) \arrow[r, "f"] & (G, \mathscr{G}) \arrow[d] \\
f^{-1}(B) \arrow[u] & B \arrow[l, "\text{preimage}"']
\end{tikzcd}$$

Random Variable

Definition (Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$.
A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is measurable, i.e.,

$$\forall B \in \mathscr{B}(\mathbb{R}), \qquad X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{preimage}"']
\end{tikzcd}$$

Probability Law of a Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

Definition (Probability Law / Distribution of X)

The **probability law** of X is the function $\mathbb{P}_X : \mathscr{B}(\mathbb{R}) \to [0, 1]$ defined by

$$\mathbb{P}_X(B) = \mathbb{P}\big(X^{-1}(B)\big), \qquad B \in \mathscr{B}(\mathbb{R}).$$

- $\mathbb{P}_X$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.
- If $E \subset \mathbb{R}$ is countable and $E = \{e_1, e_2, \ldots\}$, then

$$\mathbb{P}_X(E) = \sum_{i \in \mathbb{N}} \mathbb{P}(X = e_i).$$

Probability Law of a Random Variable (Diagram)

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{preimage}"']
\end{tikzcd}
$$

Joint Probability Law of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint Probability Law)

Given random variables $X_1, X_2, \ldots, X_n$ defined with respect to $\mathscr{F}$, their **joint probability law** is the measure

$$\mathbb{P}_{X_1,\ldots,X_n} : \mathscr{B}(\mathbb{R}^n) \to [0, 1]$$

defined by

$$\mathbb{P}_{X_1,\ldots,X_n}(B) = \mathbb{P}\big(\{\omega \in \Omega : (X_1(\omega), \ldots, X_n(\omega)) \in B\}\big), \qquad B \in \mathscr{B}(\mathbb{R}^n).$$

- $\mathbb{P}_{X_1,\ldots,X_n}$ is a probability measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$.

σ-Algebra Generated by a Random Variable

Fix a measurable space $(\Omega, \mathscr{F})$
Let $X : \Omega \to \mathbb{R}$ be a random variable.

Definition (σ-algebra generated by X)

The σ-algebra generated by X, denoted by $\sigma(X)$, is defined as

$$\sigma(X) := \left\{X^{-1}(B) : B \in \mathscr{B}(\mathbb{R})\right\}.$$

- $\sigma(X)$ is the collection of all events whose occurrence (or non-occurrence) may be decided purely based on the realization of X.
- $\sigma(X)$ is a σ-algebra of subsets of Ω.

Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $(X, Y) : \Omega \to \mathbb{R}^2$.

Definition (Independence)

We say that X and Y are independent random variables if

$$\sigma(X) \perp\!\!\!\perp \sigma(Y),$$

i.e.,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\,\mathbb{P}(A_2) \qquad \forall\, A_1 \in \sigma(X),\ A_2 \in \sigma(Y).$$

- In terms of probability laws, independence implies

$$\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1)\,\mathbb{P}_Y(B_2), \qquad \forall\, B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

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Expectation

॥ नवोन्मेषेण देशनिर्माणम् ॥
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Fix $(\Omega, \mathscr{F}, \mathbb{P})$ and let X be a random variable defined on this probability space.

Definition (Expectation)

The expectation of X is

$$\mathbb{E}[X] = \int_{\Omega} X \, d\mathbb{P} = \int_{\omega \in \Omega} X(\omega) \, d\mathbb{P}(\omega).$$

For any $A \in \mathscr{F}$,

$$\int_A X \, d\mathbb{P} = \int_{\Omega} X \, \mathbf{1}_A \, d\mathbb{P} = \int_{\Omega} X(\omega) \, \mathbf{1}_A(\omega) \, d\mathbb{P}(\omega) = \mathbb{E}[X \, \mathbf{1}_A].$$

Hence,

$$\mathbb{E}[X] = \int_A X \, d\mathbb{P} + \int_{A^c} X \, d\mathbb{P}.$$

Supremum

Let $A \subseteq \mathbb{R}$ be a subset of real numbers

- The **supremum** of the set A is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is an **upper bound** for the set A, i.e.,
  $$\forall\, y \in A, \qquad y \leq x.$$
  - For any arbitrary choice of $\varepsilon > 0$, the number $x - \varepsilon$ is not an upper bound for A (x is the **least** among all upper bounds for the set A)
  Mathematically,
  $$\forall \varepsilon > 0, \quad \exists\, y_\varepsilon \in A \quad \text{such that} \quad y_\varepsilon > x - \varepsilon.$$
  - Notation: $x = \sup A$

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers.

- The **supremum** of the sequence $\{x_n\}_{n \in \mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is an **upper bound** for the sequence, i.e., $x_n \leq x$ for all $n \in \mathbb{N}$
  - x is the **least** among all upper bounds for the sequence, i.e.,
  $$\forall \varepsilon > 0, \quad \exists\, N_\varepsilon \in \mathbb{N} \quad \text{such that} \quad x_{N_\varepsilon} > x - \varepsilon.$$

Tidbits About Supremum

- Example: suppose $A = (-2, 3)$, then $\sup A = 3$
- Supremum of a set **need not be** an element of the set
  If supremum of a set belongs to the set, it is called the **maximum**
- By convention, if $A = \emptyset$, then $\sup A = -\infty$
- In the definition of supremum,

$$\text{for every choice of } \varepsilon > 0 \qquad \Longleftrightarrow \qquad \text{for every choice of } \varepsilon \in \mathbb{Q},\ \varepsilon > 0$$

  This holds true because **rational numbers are dense in the set of real numbers**

Supremum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a sequence of random variables w.r.t. $\mathscr{F}$

Definition (Supremum of Sequence of Random Variables)

The **supremum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \sup_{n\geq 1} X_n(\omega), \qquad \omega \in \Omega.$$

- The supremum of a sequence of random variables is a random variable
- Indeed, for any $x \in \mathbb{R} \cup \{\pm\infty\}$,

$$\{X \leq x\} = \bigcap_{n\in\mathbb{N}} \{X_n \leq x\}.$$

Infimum

Let $A \subseteq \mathbb{R}$ be a subset of real numbers

- The **infimum** of the set A is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is a **lower bound** for the set A, i.e.,
$$\forall\, y \in A, \qquad y \geq x.$$
  - For any arbitrary choice of $\varepsilon > 0$, the number $x + \varepsilon$ is not a lower bound for A (x is the **greatest** among all lower bounds for the set A)
Mathematically,
$$\forall \varepsilon > 0, \quad \exists\, y_\varepsilon \in A \quad \text{such that} \quad y_\varepsilon < x + \varepsilon.$$
  - Notation: $x = \inf A$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

- The **infimum** of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is a **lower bound** for the sequence, i.e., $x_n \geq x$ for all $n \in \mathbb{N}$, and
  - x is the **greatest** among all lower bounds for the sequence, i.e.,
$$\forall \varepsilon > 0, \quad \exists\, N_\varepsilon \in \mathbb{N} \quad \text{such that} \quad x_{N_\varepsilon} < x + \varepsilon.$$

Tidbits About Infimum

- Example: suppose $A = (-2, 3)$, then $\inf A = -2$
- Infimum of a set **need not be** an element of the set
  If infimum of a set belongs to the set, then it is called the **minimum**
- By convention, if $A = \emptyset$, then $\inf A = +\infty$
- For any non-empty set A,
$$\inf A \leq \sup A.$$
- In the definition of infimum,
$$\text{for every choice of } \varepsilon > 0 \quad \Longleftrightarrow \quad \text{for every choice of } \varepsilon \in \mathbb{Q},\ \varepsilon > 0$$
  This holds true because **rational numbers are dense in the set of real numbers**

Infimum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a sequence of random variables w.r.t. $\mathscr{F}$

Definition (Infimum of a Sequence of Random Variables)

The **infimum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \inf_{n\geq 1} X_n(\omega), \qquad \omega \in \Omega.$$

- The infimum of a sequence of random variables is a random variable
- Indeed, for any $x \in \mathbb{R} \cup \{\pm\infty\}$,
$$\{X \geq x\} = \bigcap_{n\in\mathbb{N}} \{X_n \geq x\}.$$

Limit Supremum, Limit Infimum, Limit

Limit Supremum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Supremum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Supremum of a Sequence of Real Numbers

The **limit supremum** of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$x = \inf_{n\geq 1} \sup_{k\geq n} x_k; \qquad \textbf{Notation:} \quad x = \limsup_{n\to\infty} x_n.$$

- For each $n \in \mathbb{N}$, let $\quad y_n = \sup_{k\geq n} x_k \qquad x = \inf_{n\geq 1} y_n$
- If $x = \limsup_{n\to\infty} x_n$, then:
  - For every choice of primary index $n \in \mathbb{N}$, there exists a secondary index $k \geq n$ such that $x_k \geq x$ **(Infinitely many elements of the sequence are $\geq x$)**
  - For every choice of $\varepsilon > 0$, there exists an index $N_\varepsilon \in \mathbb{N}$ such that

$$x_n \leq x + \varepsilon \qquad \forall\, n \geq N_\varepsilon.$$

(All but finitely many elements of the sequence are $\leq x + \varepsilon$)

Limit Supremum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a collection of random variables w.r.t. $\mathscr{F}$

Definition (Limit Supremum of a Sequence of Random Variables)

The **limit supremum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \inf_{n\geq 1} \sup_{k\geq n} X_k(\omega), \qquad \omega \in \Omega.$$

Notation: $X = \limsup_{n\to\infty} X_n$

- **Exercise:** The limit supremum of a sequence of random variables is a random variable

Limit Infimum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Infimum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Infimum of a Sequence of Real Numbers

The **limit infimum** of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$x = \sup_{n\geq 1} \inf_{k\geq n} x_k; \qquad \textbf{Notation:} \quad x = \liminf_{n\to\infty} x_n.$$

- For each $n \in \mathbb{N}$, let $\quad y_n = \inf_{k\geq n} x_k \qquad x = \sup_{n\geq 1} y_n$
- If $x = \liminf_{n\to\infty} x_n$, then:
  - For every choice of primary index $n \in \mathbb{N}$, there exists a secondary index $k \geq n$ such that $x_k \leq x$ **(Infinitely many elements of the sequence are $\leq x$)**
  - For every choice of $\varepsilon > 0$, there exists an index $N_\varepsilon \in \mathbb{N}$ such that

$$x_n \;\geq\; x - \varepsilon \qquad \forall\, n \geq N_\varepsilon.$$

(All but finitely many elements of the sequence are $\geq x - \varepsilon$)

Limit Infimum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a collection of random variables w.r.t. $\mathscr{F}$

Definition (Limit Infimum of a Sequence of Random Variables)

The **limit infimum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \sup_{n\geq 1} \inf_{k\geq n} X_k(\omega), \qquad \omega \in \Omega.$$

Notation: $X = \liminf_{n\to\infty} X_n$

- **Exercise:** The limit infimum of a sequence of random variables is a random variable

Limit of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit of a Sequence of Real Numbers

We say that $x \in \mathbb{R} \cup \{\pm\infty\}$ is the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ if

$$\liminf_{n\to\infty} x_n = x = \limsup_{n\to\infty} x_n.$$

Equivalently, for every choice of $\varepsilon > 0$, there exists an index $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \qquad \forall\, n \geq N_\varepsilon.$$

Notation: $x = \lim_{n\to\infty} x_n.$

- The limit of a sequence, if it exists, is unique
- Not every sequence admits a limit
  However, every sequence admits a unique limit supremum and a unique limit infimum
- As before,

Limit of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Let $\{X_n\}_{n \in \mathbb{N}} = \{X_1, X_2, \ldots\}$ be a collection of random variables w.r.t. $\mathscr{F}$

- Fix $\omega \in \Omega$, and consider the sequence of real numbers

$$X_1(\omega), X_2(\omega), \ldots$$

- A limit may or may not exist for the above sequence

Lemma (An Important Set and its Measurability)

The set of all $\omega \in \Omega$ for which $\lim_{n \to \infty} X_n(\omega)$ exists is a valid event, i.e.,

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n \to \infty} X_n(\omega) \text{ exists} \right\} \in \mathscr{F}.$$

Proof of Lemma 1

Monotone Convergence Theorem

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X, X_1, X_2, \ldots$ be nonnegative random variables.

Theorem (Monotone Convergence Theorem)

Suppose that

$$0 \leq X_1(\omega) \leq X_2(\omega) \leq \cdots \qquad \forall \omega \in \Omega$$

and $X_n(\omega) \to X(\omega)$ pointwise as $n \to \infty$. Then

$$\mathbb{E}[X] = \lim_{n \to \infty} \mathbb{E}[X_n].$$