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Conditional Expectation (over a σ-algebra)

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Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $X : \Omega \to \mathbb{R}$ be an $\mathscr{F}$-measurable random variable.

Now let $\mathscr{G} \subseteq \mathscr{F}$ be a sub-σ-algebra.

Definition (Conditional Expectation)

A random variable Y is called a conditional expectation of X given $\mathscr{G}$, denoted $Y = \mathbb{E}[X \mid \mathscr{G}]$, if:

1. Y is $\mathscr{G}$-measurable,
2. X and Y are integrable, and
3. for every $G \in \mathscr{G}$,

$$\int_G Y \, d\mathbb{P} = \int_G X \, d\mathbb{P}.$$

- Existence and uniqueness of $\mathbb{E}[X \mid \mathscr{G}]$ (a.s.) follow from the Radon–Nikodym theorem

Conditional Probability (over a σ-algebra)

For any event $A \in \mathscr{F}$, define the indicator random variable

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \text{else.} \end{cases}$$

Definition (Conditional Probability)

The **conditional probability** of A given $\mathscr{G}$ is defined as

$$\mathbb{P}(A \mid \mathscr{G}) := \mathbb{E}[\mathbf{1}_A \mid \mathscr{G}].$$

Conditional Probability: Characterization

Now, define

$$\mathbb{P}(A \mid \mathscr{G}) := \mathbb{E}[\mathbf{1}_A \mid \mathscr{G}],$$

where $\mathbb{P}(A \mid \mathscr{G})$ is a $\mathscr{G}$-measurable random variable.

Then, for every $G \in \mathscr{G}$,

$$\int_G \mathbb{P}(A \mid \mathscr{G})\, d\mathbb{P} = \int_G \mathbf{1}_A \, d\mathbb{P} = \int_\Omega \mathbf{1}_G \, \mathbf{1}_A \, d\mathbb{P} = \int_\Omega \mathbf{1}_{A \cap G} \, d\mathbb{P} = \mathbb{P}(A \cap G).$$

Conditional Probability: Discrete Case

Let $\{B_i\}_{i \in I}$ be a partition of Ω, i.e.,

- $B_i \cap B_j = \emptyset$ for $i \neq j$,
- $\bigcup_{i \in I} B_i = \Omega$.

Define

$$\mathscr{G} = \sigma(B_1, B_2, \ldots).$$

Conditional Probability: Discrete Case (Derivation)

Suppose Y is a $\mathscr{G}$-measurable random variable. For each j, define

$$c_j := Y(\omega) \qquad \text{for any } \omega \in B_j.$$

Then for $\omega_1, \omega_2 \in B_j$, we have $Y(\omega_1) = Y(\omega_2)$, and hence

$$Y(\omega) = \sum_i c_i \, \mathbf{1}_{B_i}(\omega).$$

In particular, any $\mathscr{G}$-measurable function has the form $\sum_i c_i \mathbf{1}_{B_i}(\omega)$, so

$$\mathbb{P}(A \mid \mathscr{G})(\omega) = \sum_i c_i \, \mathbf{1}_{B_i}(\omega).$$

Consider, for any $B_j \in \mathscr{G}$,

$$\int_{B_j} \mathbb{P}(A \mid \mathscr{G}) \, d\mathbb{P} = \int_{B_j} \mathbf{1}_A \, d\mathbb{P} = \int_{\Omega} \mathbf{1}_{A \cap B_j} \, d\mathbb{P} = \mathbb{P}(A \cap B_j).$$

Conditional Probability: Discrete Case (Formula)

From

$$\int_{B_j} \mathbb{P}(A \mid \mathscr{G})\, d\mathbb{P} = \int_{B_j} \sum_i c_i\, \mathbf{1}_{B_i}(\omega)\, d\mathbb{P} = c_j \int_{B_j} d\mathbb{P} = c_j\, \mathbb{P}(B_j),$$

we get

$$c_j = \frac{\mathbb{P}(A \cap B_j)}{\mathbb{P}(B_j)} = \mathbb{P}(A \mid B_j).$$

Therefore,

$$\mathbb{P}(A \mid \mathscr{G})(\omega) = \sum_i \mathbb{P}(A \mid B_i)\, \mathbf{1}_{B_i}(\omega) = \begin{cases} \mathbb{P}(A \mid B_1), & \omega \in B_1, \\ \mathbb{P}(A \mid B_2), & \omega \in B_2, \\ \vdots \end{cases}$$

Properties of Conditional Expectation (1/2)

Let Y be integrable and let $\mathscr{G} \subseteq \mathscr{F}$ be a sub-σ-field.

Basic facts

$\mathbb{E}[Y \mid \mathscr{G}]$ is a $\mathscr{G}$-measurable random variable such that

$$\int_A \mathbb{E}[Y \mid \mathscr{G}]\, d\mathbb{P} = \int_A Y\, d\mathbb{P} \qquad \forall\, A \in \mathscr{G}.$$

1. **Linearity:** for integrable Y_1, Y_2 and scalars a, b,

$$\mathbb{E}[aY_1 + bY_2 \mid \mathscr{G}] = a\,\mathbb{E}[Y_1 \mid \mathscr{G}] + b\,\mathbb{E}[Y_2 \mid \mathscr{G}].$$

2. **Pull-out property:** if Z is $\mathscr{G}$-measurable and Y integrable,

$$\mathbb{E}[ZY \mid \mathscr{G}] = Z\,\mathbb{E}[Y \mid \mathscr{G}].$$

Properties of Conditional Expectation (2/2)

3. **Positivity and monotonicity:**
   - If $Y \geq 0$, then $\mathbb{E}[Y \mid \mathscr{G}] \geq 0$.
   - If $Y_1 \leq Y_2$, then $\mathbb{E}[Y_1 \mid \mathscr{G}] \leq \mathbb{E}[Y_2 \mid \mathscr{G}]$.
4. **Tower property:** if $\mathscr{H} \subseteq \mathscr{G}$ are σ-fields and Y is integrable, then

$$\mathbb{E}[\,\mathbb{E}[Y \mid \mathscr{G}] \mid \mathscr{H}] = \mathbb{E}[Y \mid \mathscr{H}].$$

(almost surely)

Tower Property (Iterated Conditional Expectation)

Statement

For a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $\mathscr{H}$ and $\mathscr{G}$ be sub-σ-fields with $\mathscr{H} \subseteq \mathscr{G} \subseteq \mathscr{F}$. If X is integrable (i.e., $\mathbb{E}[|X|] < \infty$), then

$$\mathbb{E}\big[\,\mathbb{E}[X \mid \mathscr{G}] \mid \mathscr{H}\,\big] = \mathbb{E}[X \mid \mathscr{H}] \qquad \text{a.s.}$$

Let $Y := \mathbb{E}[X \mid \mathscr{G}]$, $W := \mathbb{E}[Y \mid \mathscr{H}]$, $Z := \mathbb{E}[X \mid \mathscr{H}]$.

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Tower Property: Proof Sketch

By definition of conditional expectation:

- Y is $\mathscr{G}$-measurable and $\int_G Y\, d\mathbb{P} = \int_G X\, d\mathbb{P}$ for all $G \in \mathscr{G}$.
- Z is $\mathscr{H}$-measurable and $\int_H Z\, d\mathbb{P} = \int_H X\, d\mathbb{P}$ for all $H \in \mathscr{H}$.
- W is $\mathscr{H}$-measurable and $\int_H W\, d\mathbb{P} = \int_H Y\, d\mathbb{P}$ for all $H \in \mathscr{H}$.

Since $\mathscr{H} \subseteq \mathscr{G}$, for any $H \in \mathscr{H}$ we also have $H \in \mathscr{G}$, hence

$$\int_H Y\, d\mathbb{P} = \int_H X\, d\mathbb{P}.$$

Combining with the defining property of W gives

$$\int_H W\, d\mathbb{P} = \int_H X\, d\mathbb{P}.$$

But Z is the (a.s.-unique) $\mathscr{H}$-measurable random variable with this property, therefore $W = Z$ a.s.