Kolmogorov 0-1 Law

Let $X_1, X_2, \ldots$ be a sequence of independent random variables.

Statement (Kolmogorov 0-1 law)

Any event A in the tail σ-field

$$\mathscr{G}^\infty := \bigcap_{i=1}^{\infty} \sigma(X_i, X_{i+1}, X_{i+2}, \ldots)$$

has probability 0 or 1, i.e., $\mathbb{P}(A) \in \{0, 1\}$.

(Tail event / tail field.)

Tail σ-field: Expansion

$$\bigcap_{i=1}^{\infty} \sigma(X_i, X_{i+1}, X_{i+2}, \ldots) = \sigma(X_1, X_2, \ldots) \cap \sigma(X_2, X_3, \ldots) \cap \sigma(X_3, X_4, \ldots) \cap \cdots$$

Idea (sketch) Behind Kolmogorov 0-1 Law

Let A be an event.

- If $A \in \sigma(X_m, X_{m+1}, \dots)$, then

$$A \in \sigma(X_1, X_2, \dots) \quad \Rightarrow \quad \sigma(X_m, X_{m+1}, \dots) \subseteq \sigma(X_1, \dots, X_m, X_{m+1}, \dots).$$

- Define the tail field

$$\mathscr{G}^\infty = \bigcap_{m=1}^{\infty} \sigma(X_m, X_{m+1}, \dots).$$

- If $A \in \mathscr{G}^\infty$, then it also belongs to $\mathscr{G} := \sigma(X_1, X_2, \dots)$, hence $\mathscr{G}^\infty \subseteq \mathscr{G}$.
- Key point: one can show A is independent of $\sigma(X_1, X_2, X_3, \dots)$.

If $A \perp\!\!\!\perp \sigma(X_1, X_2, \dots)$ and also $A \in \sigma(X_1, X_2, \dots)$, then

$$\mathbb{P}(A \cap A) = \mathbb{P}(A)\,\mathbb{P}(A) \quad \Rightarrow \quad \mathbb{P}(A) = \mathbb{P}(A)^2 \quad \Rightarrow \quad \mathbb{P}(A) \in \{0, 1\}.$$

Appendix: Strong Law of Large Numbers (SLLN)

Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $\mathbb{E}[|X_1|] < \infty$.

Statement (SLLN)

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \mathbb{E}[X_1].$$

(If not identical, only independent: one may use Kolmogorov's three-series theorem.)

SLLN as a Tail Event (sketch)

Fix any $m \in \mathbb{N}$. Then trivially

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{m} X_i = 0.$$

Define

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \text{ exists} \right\}.$$

Since finite prefixes do not affect the existence of the limit,

$$A_{\lim} \in \bigcap_{i=m+1}^{\infty} \sigma(X_i, X_{i+1}, \dots) \quad \text{for every } m,$$

hence $A_{\lim}$ is a tail event.

SLLN as a Tail Event (continued)

By Kolmogorov 0-1 law, $\mathbb{P}(A_{\text{lim}}) \in \{0, 1\}$.
Moreover, from the definition of almost sure convergence, one expects

$$\mathbb{P}(A_{\text{lim}}) = 1.$$