Filtration

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition (Filtration)

A **filtration** is a sequence of σ-algebras

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \cdots \subseteq \mathscr{F}_n \subseteq \cdots$$

such that

1. each $\mathscr{F}_n$ is a σ-algebra,
2. the sequence is increasing: $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$.

- $\mathscr{F}_n$ represents the set of all events that are known/observable at time n.
- (Remark) A *natural filtration* generated by a process (X_n) is $\mathscr{F}_n = \sigma(X_0, \ldots, X_n)$.

Stochastic Process and Adaptedness

A **stochastic process** is a collection of random variables

$$\{X_t\}_{t \in T},$$

where T is an index set (often interpreted as time).

Definition (Adapted Stochastic Process)

Given a filtration $\{\mathscr{F}_n\}$, a process $\{X_n\}$ is said to be **adapted** to $\{\mathscr{F}_n\}$ if, for every n,

$$X_n \text{ is } \mathscr{F}_n\text{-measurable} \iff X_n^{-1}(B) \in \mathscr{F}_n \ \ \forall B \in \mathscr{B}(\mathbb{R}).$$

Intuitively, X_n may depend on the past observations $(X_0, \ldots, X_n)$, but not on the future.

Example: Filtration from Coin Tosses

Consider a sequence of random variables X_1, X_2, X_3 on $(\Omega, \mathscr{F}, \mathbb{P})$ generated by three coin tosses.

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

- At time $T = 0$ (before observing anything):
$$\mathscr{F}_0 = \{\emptyset, \Omega\}.$$
- At time $T = 1$ (you only observe the first toss):
$$\mathscr{F}_1 = \{\emptyset, \Omega, A_H, A_T\},$$
where $A_H := \{HHH, HHT, HTH, HTT\}$ and $A_T := \{THH, THT, TTH, TTT\}$.

Example: Filtration from Coin Tosses

At time $T = 2$ (you observe the first two tosses), define the four events

$$A_{HH} = \{HHH, HHT\}, \quad A_{HT} = \{HTH, HTT\}, \quad A_{TH} = \{THH, THT\}, \quad A_{TT} = \{TTH, TTT\}.$$

Then $\mathscr{F}_2$ is the σ-algebra generated by this partition:

$$\mathscr{F}_2 = \sigma(A_{HH}, A_{HT}, A_{TH}, A_{TT}),$$

i.e., it contains $\emptyset$, Ω, these four sets, and all unions and complements built from them.

- We can observe that $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3 = \mathscr{F}$.
- $\mathscr{F}_{n+1}$ contains (at least) as much information as $\mathscr{F}_n$.

Martingales

Let $X_1, X_2, \ldots$ be random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\{\mathscr{F}_n\}_{n\geq 1}$ be a sequence of σ-fields in $\mathscr{F}$.

Definition (Martingale)

The sequence $\{(X_n, \mathscr{F}_n)\}_{n\geq 1}$ is a **martingale** if the following hold for all n:

1. Filtration: $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$.
2. Adaptedness: X_n is $\mathscr{F}_n$-measurable.
3. Integrability: $\mathbb{E}[|X_n|] < \infty$.
4. Martingale property:

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n \qquad \text{a.s.}$$

- For a **submartingale**: $\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] \geq X_n$ a.s.
- For a **supermartingale**: $\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] \leq X_n$ a.s.

Martingales (Interpretation)

- Interpretation: a martingale is a *fair game*: the conditional expected next value equals the current value.
- Natural filtration: $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$ is the smallest filtration to which (X_n) is adapted.
- One may also allow a larger filtration (more information) than the natural filtration.

Equivalent Integral Formulation

The martingale property

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n$$

is equivalent (by the defining property of conditional expectation) to the integral identity:

$$\int_A X_{n+1} \, d\mathbb{P} = \int_A X_n \, d\mathbb{P} \qquad \forall A \in \mathscr{F}_n.$$

(Recall: $Y = \mathbb{E}[X \mid \mathscr{G}]$ iff Y is $\mathscr{G}$-measurable and $\int_G Y \, d\mathbb{P} = \int_G X \, d\mathbb{P}$ for all $G \in \mathscr{G}$.)

Alternative Formulation (Increments)

Define the **gain/increment**

$$\Delta_n := X_n - X_{n-1}, \qquad \text{with } \Delta_1 := X_1.$$

Then

$$\mathbb{E}[X_{n+1} - X_n \mid \mathscr{F}_n] = \mathbb{E}[X_{n+1} \mid \mathscr{F}_n] - \mathbb{E}[X_n \mid \mathscr{F}_n] = \mathbb{E}[X_{n+1} \mid \mathscr{F}_n] - X_n = 0.$$

Interpretation: expected future wealth equals present wealth.

Multistep-ahead Property

If $\{(X_n, \mathscr{F}_n)\}$ is a martingale, then for every $k > n$,

$$X_n = \mathbb{E}[X_k \mid \mathscr{F}_n].$$

Proof idea:

$$X_n = \mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = \mathbb{E}\big[\,\mathbb{E}[X_{n+2} \mid \mathscr{F}_{n+1}] \mid \mathscr{F}_n\big] = \mathbb{E}[X_{n+2} \mid \mathscr{F}_n] = \cdots = \mathbb{E}[X_{n+k} \mid \mathscr{F}_n].$$

Constant Expectation

If $\{(X_n, \mathscr{F}_n)\}$ is a martingale, then $\mathbb{E}[X_n]$ is constant in n.

Proof: take expectations on both sides of $\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n$:

$$\mathbb{E}[X_n] = \mathbb{E}\left[\mathbb{E}[X_{n+1} \mid \mathscr{F}_n]\right] = \mathbb{E}[X_{n+1}].$$

Consequently:

- for submartingales, $\mathbb{E}[X_{n+1}] \geq \mathbb{E}[X_n]$ (monotone increasing),
- for supermartingales, $\mathbb{E}[X_{n+1}] \leq \mathbb{E}[X_n]$ (monotone decreasing).

Convex Functions of Martingales (Jensen)

Let $\{(X_n, \mathscr{F}_n)\}_{n\geq 1}$ be a martingale and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. Assume $\varphi(X_n)$ is integrable for each n.

Theorem

The process $\{(\varphi(X_n), \mathscr{F}_n)\}_{n\geq 1}$ is a **submartingale**:

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathscr{F}_n] \geq \varphi(X_n) \qquad \text{a.s.}$$

Proof (Conditional Jensen / Tower)

By conditional Jensen's inequality (for convex φ and integrable X_{n+1}),

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathscr{F}_n] \geq \varphi\big(\mathbb{E}[X_{n+1} \mid \mathscr{F}_n]\big).$$

Using the martingale property $\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n$ a.s., we get

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathscr{F}_n] \geq \varphi(X_n),$$

hence $\{\varphi(X_n)\}$ is a submartingale.

Convex Functions of Submartingales

Let $\{(X_n, \mathscr{F}_n)\}_{n\geq 1}$ be a submartingale and let $\varphi : \mathbb{R} \to \mathbb{R}$ be **nondecreasing** and convex. Assume $\varphi(X_n)$ is integrable for each n.

Theorem

The process $\{(\varphi(X_n), \mathscr{F}_n)\}_{n\geq 1}$ is a **submartingale**:

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathscr{F}_n] \geq \varphi(X_n) \qquad \text{a.s.}$$

Proof (Monotonicity + Conditional Jensen)

Since $\{X_n\}$ is a submartingale,

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] \geq X_n \qquad \text{a.s.}$$

and because φ is nondecreasing,

$$\varphi\big(\mathbb{E}[X_{n+1} \mid \mathscr{F}_n]\big) \geq \varphi(X_n) \qquad \text{a.s.}$$

By conditional Jensen's inequality,

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathscr{F}_n] \geq \varphi\big(\mathbb{E}[X_{n+1} \mid \mathscr{F}_n]\big).$$

Combining gives $\mathbb{E}[\varphi(X_{n+1}) \mid \mathscr{F}_n] \geq \varphi(X_n)$ a.s.

Note: if φ is decreasing (e.g. $\varphi(x) = -x$), then a submartingale can become a **supermartingale**.

Doob's Martingale

Let $X_1, X_2, \ldots$ be integrable random variables and let $\mathscr{F}_n := \sigma(X_1, \ldots, X_n)$ be the natural filtration.

Fix any integrable random variable Y (often $Y = f(X_1, \ldots, X_N)$ for some N).

Definition (Doob's martingale)

Define

$$M_n := \mathbb{E}[Y \mid \mathscr{F}_n], \qquad n \geq 1.$$

Then $\{(M_n, \mathscr{F}_n)\}_{n\geq 1}$ is a martingale.

Doob's Martingale: Proof (Tower Property)

Let $M_n := \mathbb{E}[Y \mid \mathscr{F}_n]$.

- **Adaptedness:** M_n is $\mathscr{F}_n$-measurable by definition.
- **Integrability:** $\mathbb{E}[|M_n|] \leq \mathbb{E}[|Y|] < \infty$.
- **Martingale property:** for $n \geq 1$,

$$\mathbb{E}[M_{n+1} \mid \mathscr{F}_n] = \mathbb{E}\big[\mathbb{E}[Y \mid \mathscr{F}_{n+1}] \mid \mathscr{F}_n\big] = \mathbb{E}[Y \mid \mathscr{F}_n] = M_n,$$

where the middle equality is the **tower property** since $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$.

Stopping Time

Let $\tau : \Omega \to \{1, 2, 3, \ldots\} \cup \{\infty\}$ be a random time.

Definition (Stopping time)

We say τ is a **stopping time** with respect to a filtration $\{\mathscr{F}_n\}$ if

$$\{\tau = k\} \in \mathscr{F}_k \qquad \text{for each finite } k.$$

Equivalently (discrete-time),

$$\{\tau \leq k\} \in \mathscr{F}_k \qquad \text{for each finite } k.$$

Stopping Time: Equivalent Characterizations (1/2)

Claim (discrete-time):

$$\{\tau = k\} \in \mathscr{F}_k \ \forall k \quad \Longleftrightarrow \quad \{\tau \le n\} \in \mathscr{F}_n \ \forall n.$$

Proof: assume $\{\tau = k\} \in \mathscr{F}_k$ for all k. Then for any n,

$$\{\tau \le n\} = \bigcup_{k=1}^{n} \{\tau = k\} \in \mathscr{F}_n,$$

since $\mathscr{F}_k \subseteq \mathscr{F}_n$ for $k \le n$.

Stopping Time: Equivalent Characterizations (2/2)

Proof (converse): assume $\{\tau \le n\} \in \mathscr{F}_n$ for all n. Then

$$\{\tau = n\} = \{\tau \le n\} \setminus \{\tau \le n-1\} \in \mathscr{F}_n.$$

Remark: in continuous-time, the two definitions require additional care.

Example: Stopping Time (Hitting Time)

Let $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$ and fix a threshold $a \in \mathbb{R}$. Define the (first) hitting time

$$\tau := \inf\{n \geq 1 : X_n \geq a\} \in \{1, 2, \ldots\} \cup \{\infty\}.$$

For any $k \geq 1$,

$$\{\tau = k\} = \{X_1 < a, \ldots, X_{k-1} < a,\ X_k \geq a\} \in \mathscr{F}_k.$$

Hence τ is a stopping time.

Remark: this is also called a **hitting time**.

The σ-algebra $\mathscr{F}_\tau$

Let τ be a stopping time w.r.t. $\{\mathscr{F}_n\}$.

Definition (σ-algebra at time τ)

$$\mathscr{F}_\tau := \{A \subseteq \Omega : A \cap \{\tau \leq k\} \in \mathscr{F}_k \text{ for all } k \geq 1\}.$$

Intuition: events in $\mathscr{F}_\tau$ are those whose occurrence can be decided using information available *up to the random time* τ.

Example: Coin Tosses and $\mathscr{F}_\tau$

Consider three coin tosses with

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Define $X_n(\omega)$ as the result of the nth toss (H or T), and the natural filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \quad \mathscr{F}_1 = \sigma(X_1), \quad \mathscr{F}_2 = \sigma(X_1, X_2), \quad \mathscr{F}_3 = \sigma(X_1, X_2, X_3) = 2^\Omega.$$

Define the stopping time "first head (or 3 if no head)" by

$$\tau(\omega) = \begin{cases} 1, & X_1(\omega) = H, \\ 2, & X_1(\omega) = T,\ X_2(\omega) = H, \\ 3, & X_1(\omega) = T,\ X_2(\omega) = T. \end{cases}$$

Verifying τ and Computing $\mathscr{F}_\tau$

For example,

$$\{\tau \leq 2\} = \{X_1 = H\} \cup \{X_1 = T, X_2 = H\} \in \mathscr{F}_2,$$

hence τ is a stopping time.

The values of τ partition Ω into

$$H := \{HHH, HHT, HTH, HTT\} \qquad (\tau = 1),$$
$$TH := \{THH, THT\} \qquad (\tau = 2),$$
$$TT := \{TTH, TTT\} \qquad (\tau = 3).$$

Therefore,

$$\mathscr{F}_\tau = \sigma(H, TH, TT).$$

Stopped Process

Let $\{X_n\}$ be a stochastic process and τ a stopping time.

Definition (Stopped process)

The **stopped process** is

$$X_n^\tau := X_{n\wedge\tau}, \qquad n \geq 1.$$

Interpretation: once the time τ occurs, the process is ”frozen” at X_τ.

Stopped Martingale

Let $\{(X_n, \mathscr{F}_n)\}_{n\geq 1}$ be a martingale and let τ be a stopping time. Define the stopped process

$$X_n^* := X_{n\wedge\tau} = \begin{cases} X_n, & \tau \geq n, \\ X_\tau, & \tau < n. \end{cases}$$

Theorem

The process $\{(X_n^*, \mathscr{F}_n)\}_{n\geq 1}$ is a martingale.

Proof: Integrability of Stopped Martingale

For any n,

$$\mathbb{E}[|X_n^*|] = \int_\Omega |X_n^*| \, d\mathbb{P} = \int_{\{\tau<n\}} |X_n^*| \, d\mathbb{P} + \int_{\{\tau\geq n\}} |X_n^*| \, d\mathbb{P}.$$

Since $\{\tau < n\} = \bigcup_{k=1}^{n-1} \{\tau = k\}$,

$$\mathbb{E}[|X_n^*|] = \sum_{k=1}^{n-1} \int_{\{\tau=k\}} |X_k| \, d\mathbb{P} + \int_{\{\tau\geq n\}} |X_n| \, d\mathbb{P} \leq \sum_{k=1}^{n-1} \mathbb{E}[|X_k|] + \mathbb{E}[|X_n|] < \infty.$$

Proof: Adaptedness of Stopped Martingale

Fix n and $B \in \mathscr{B}(\mathbb{R})$. Then

$$\{X_n^* \in B\} = \left(\bigcup_{k=1}^{n-1} (\{\tau = k\} \cap \{X_k \in B\}) \right) \cup (\{\tau \geq n\} \cap \{X_n \in B\}).$$

Here $\{\tau = k\} \in \mathscr{F}_k \subseteq \mathscr{F}_n$, $\{X_k \in B\} \in \mathscr{F}_k \subseteq \mathscr{F}_n$, and $\{\tau \geq n\} = \{\tau \leq n-1\}^c \in \mathscr{F}_n$, hence $\{X_n^* \in B\} \in \mathscr{F}_n$.

Proof: Martingale Property

Let $A \in \mathscr{F}_n$. We show $\int_A X^*_{n+1}\, d\mathbb{P} = \int_A X^*_n\, d\mathbb{P}$.

Using $X^*_n = X_n \mathbf{1}_{\{\tau \geq n\}} + X_\tau \mathbf{1}_{\{\tau < n\}}$ and $X^*_{n+1} = X_{n+1} \mathbf{1}_{\{\tau > n\}} + X_\tau \mathbf{1}_{\{\tau \leq n\}}$,

$$\int_A X^*_n\, d\mathbb{P} = \int_{A \cap \{\tau \geq n\}} X_n\, d\mathbb{P} + \int_{A \cap \{\tau < n\}} X_\tau\, d\mathbb{P},$$

$$\int_A X^*_{n+1}\, d\mathbb{P} = \int_{A \cap \{\tau > n\}} X_{n+1}\, d\mathbb{P} + \int_{A \cap \{\tau \leq n\}} X_\tau\, d\mathbb{P}.$$

Proof: Martingale Property

Note that $A \cap \{\tau > n\} \in \mathscr{F}_n$ because $A \in \mathscr{F}_n$ and $\{\tau > n\} = \{\tau \leq n\}^c \in \mathscr{F}_n$.

Since $\{X_n\}$ is a martingale,

$$\int_{A\cap\{\tau>n\}} X_{n+1}\, d\mathbb{P} = \int_{A\cap\{\tau>n\}} X_n\, d\mathbb{P}.$$

Proof: Martingale Property

Decompose $A \cap \{\tau \geq n\} = (A \cap \{\tau > n\}) \,\dot{\cup}\, (A \cap \{\tau = n\})$ and $A \cap \{\tau \leq n\} = (A \cap \{\tau < n\}) \,\dot{\cup}\, (A \cap \{\tau = n\})$.

Plugging these into the expressions for $\int_A X_n^*$ and $\int_A X_{n+1}^*$ shows the X_τ-terms match, and with $\int_{A\cap\{\tau>n\}} X_{n+1}\, d\mathbb{P} = \int_{A\cap\{\tau>n\}} X_n\, d\mathbb{P}$ we get

$$\int_A X_{n+1}^*\, d\mathbb{P} = \int_A X_n^*\, d\mathbb{P} \qquad \forall A \in \mathscr{F}_n.$$

Therefore $\mathbb{E}[X_{n+1}^* \mid \mathscr{F}_n] = X_n^*$ a.s.

Optional Sampling Theorem (Discrete-Time)

Let $\{(X_k, \mathscr{F}_k)\}_{k=0}^{n}$ be a **submartingale** and let τ_1, τ_2 be stopping times satisfying

$$1 \leq \tau_1 \leq \tau_2 \leq n.$$

Statement

The sampled process is again a submartingale (with respect to $\mathscr{F}_{\tau_1}, \mathscr{F}_{\tau_2}$):

$$\mathbb{E}[X_{\tau_2} \mid \mathscr{F}_{\tau_1}] \geq X_{\tau_1} \qquad \text{a.s.}$$

Equivalently, for every $A \in \mathscr{F}_{\tau_1}$,

$$\int_A X_{\tau_2}\, d\mathbb{P} \geq \int_A X_{\tau_1}\, d\mathbb{P}.$$

Proof Idea (Increments)

Define increments $\Delta_k := X_k - X_{k-1}$ for $k \geq 1$. Then

$$X_{\tau_2} - X_{\tau_1} = \sum_{k=1}^{n} \mathbf{1}_{\{\tau_1 < k \leq \tau_2\}} \Delta_k.$$

Let $A \in \mathscr{F}_{\tau_1}$. Integrating and using linearity,

$$\int_A (X_{\tau_2} - X_{\tau_1})\, d\mathbb{P} = \sum_{k=1}^{n} \int_{A \cap \{\tau_1 < k \leq \tau_2\}} \Delta_k \, d\mathbb{P}.$$

Proof (Key Measurability + Submartingale Property)

For each fixed k, the set

$$B_k := A \cap \{\tau_1 < k \leq \tau_2\}$$

belongs to $\mathscr{F}_{k-1}$ (hence to $\mathscr{F}_k$):

- $A \cap \{\tau_1 \leq k-1\} \in \mathscr{F}_{k-1}$ by definition of $\mathscr{F}_{\tau_1}$,
- $\{\tau_2 \geq k\} = \{\tau_2 \leq k-1\}^c \in \mathscr{F}_{k-1}$ since τ_2 is a stopping time.

Hence $B_k \in \mathscr{F}_{k-1}$.

Since $\{X_k\}$ is a submartingale, for any $B \in \mathscr{F}_{k-1}$ we have $\int_B \Delta_k \, d\mathbb{P} = \int_B (X_k - X_{k-1}) \, d\mathbb{P} \geq 0$. Therefore each term $\int_{B_k} \Delta_k \, d\mathbb{P} \geq 0$, and summing yields

$$\int_A X_{\tau_2} \, d\mathbb{P} \geq \int_A X_{\tau_1} \, d\mathbb{P}.$$

This proves $\mathbb{E}[X_{\tau_2} \mid \mathscr{F}_{\tau_1}] \geq X_{\tau_1}$ a.s.

Doob's Upcrossing Lemma: Upcrossings

Let $[\alpha, \beta]$ be an interval with $\alpha < \beta$, and let $X_1, \ldots, X_n$ be random variables.

Upcrossing times of $[\alpha, \beta]$

Define the sequence $(\tau_k)_{k \geq 1}$ by

$$\tau_1 := \inf\{1 \leq j \leq n : X_j \leq \alpha\} \wedge n,$$

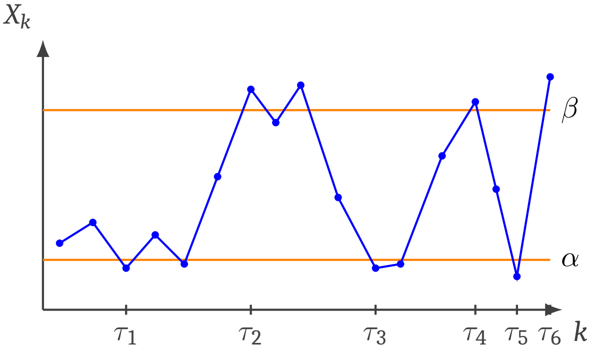
and for $k \geq 2$,

$$\tau_k := \begin{cases} \inf\{\tau_{k-1} < j \leq n : X_j \geq \beta\} \wedge n, & k \text{ even}, \\ \inf\{\tau_{k-1} < j \leq n : X_j \leq \alpha\} \wedge n, & k \text{ odd}. \end{cases}$$

The number of upcrossings of $[\alpha, \beta]$ by $X_1, \ldots, X_n$ is

$$U := \max\{i : X_{\tau_{2i-1}} \leq \alpha < \beta < X_{\tau_{2i}}\}.$$

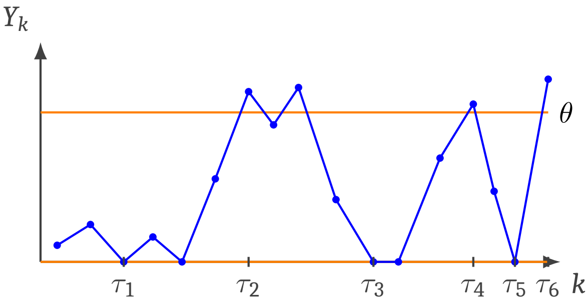
Upcrossings: Illustration (Process X_k)



Define

$$Y_k := \max\{0, X_k - \alpha\}, \qquad \theta := \beta - \alpha.$$

Upcrossings: Illustration (Process Y_k)



Recall

$$Y_k = (X_k - \alpha)^+ := \max\{0, X_k - \alpha\},$$

Upcrossings After Transformation (1/2)

Recall $Y_k = (X_k - \alpha)^+ = \max\{0, X_k - \alpha\}$ and $\theta = \beta - \alpha$.

Transformed upcrossing times of $[0, \theta]$ for (Y_k)

Define

$$\tau_1 := \inf\{j : Y_j = 0\} \wedge n,$$

and for $k \geq 2$,

$$\tau_k := \begin{cases} \inf\{j : Y_j \geq \theta,\ \tau_{k-1} < j \leq n\} \wedge n, & k \text{ even}, \\ \inf\{j : Y_j = 0,\ \tau_{k-1} < j \leq n\} \wedge n, & k \text{ odd}. \end{cases}$$

Then the upcrossings of $[\alpha, \beta]$ by (X_k) correspond to upcrossings of $[0, \theta]$ by (Y_k).

Moreover, (Y_k) is a submartingale whenever (X_k) is a submartingale, since $x \mapsto (x - \alpha)^+$ is nondecreasing and convex.

Upcrossings After Transformation (2/2)

Consider

$$Y_{\tau_n} - Y_{\tau_1} = (Y_{\tau_2} - Y_{\tau_1}) + (Y_{\tau_3} - Y_{\tau_2}) + \cdots + (Y_{\tau_n} - Y_{\tau_{n-1}}).$$

Define the even- and odd-indexed sums

$$\Sigma_e := (Y_{\tau_2} - Y_{\tau_1}) + (Y_{\tau_4} - Y_{\tau_3}) + \cdots = \sum_{\substack{k \geq 2 \\ k \text{ even}}} (Y_{\tau_k} - Y_{\tau_{k-1}}),$$

and

$$\Sigma_o := (Y_{\tau_3} - Y_{\tau_2}) + (Y_{\tau_5} - Y_{\tau_4}) + \cdots = \sum_{\substack{k \geq 3 \\ k \text{ odd}}} (Y_{\tau_k} - Y_{\tau_{k-1}}).$$

Note that $Y_{\tau_n} = Y_n$ and $Y_{\tau_1} = 0$, hence

$$Y_n = \Sigma_e + \Sigma_o. \tag{1}$$

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Upcrossing Bound via Expectations (1/2)

॥ नवोन्मेषेण देशनिर्माणम् ॥
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Applying expectation to (1) gives

$$\mathbb{E}[Y_n] = \mathbb{E}[\Sigma_e] + \mathbb{E}[\Sigma_o].$$

Even though $\Sigma_o = \sum_{\substack{k\geq 3\\ k \text{ odd}}} (Y_{\tau_k} - Y_{\tau_{k-1}})$ can contain negative increments, by optional sampling (applied to the submartingale (Y_k)) we have

$$\mathbb{E}[\Sigma_e] \geq 0, \qquad \mathbb{E}[\Sigma_o] \geq 0.$$

Hence

$$\mathbb{E}[Y_n] \geq \mathbb{E}[\Sigma_e].$$

Each upcrossing of $[0, \theta]$ contributes at least θ to Σ_e, and there are U such upcrossings, so

$$\Sigma_e \geq U\,\theta.$$

Upcrossing Bound via Expectations (2/3)

Since $Y_n = (X_n - \alpha)^+$,

$$Y_n(\omega) = (X_n(\omega) - \alpha)\,\mathbf{1}_{\{X_n > \alpha\}}(\omega).$$

Taking expectations (step by step):

$$\begin{aligned}
\mathbb{E}[Y_n] &= \int_\Omega Y_n \, d\mathbb{P} \\
&= \int_\Omega (X_n - \alpha)\,\mathbf{1}_{\{X_n > \alpha\}} \, d\mathbb{P} \\
&= \int_{\{X_n > \alpha\}} (X_n - \alpha) \, d\mathbb{P} \\
&\leq \int_{\{X_n > \alpha\}} (|X_n| + |\alpha|) \, d\mathbb{P} \\
&\leq \int (|X_n| + |\alpha|) \, d\mathbb{P}.
\end{aligned}$$

Upcrossing Bound via Expectations (3/3)

From (2) and (3) (with $\theta = \beta - \alpha$),

$$(\beta - \alpha)\, \mathbb{E}[U] \leq \mathbb{E}[|X_n|] + |\alpha|,$$

hence

$$\boxed{\mathbb{E}[U] \leq \frac{\mathbb{E}[|X_n|] + |\alpha|}{\beta - \alpha}.}$$

Fatou's Lemma

Fatou's Lemma

Let $(Y_n)_{n\geq 1}$ be a sequence of nonnegative random variables. Then

$$\mathbb{E}\left[\liminf_{n\to\infty} Y_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[Y_n].$$

Martingale Convergence Theorem

Theorem (Martingale Convergence Theorem)

Let $\{X_n\}_{n\geq 1}$ be a submartingale. If

$$K := \sup_{n\geq 1} \mathbb{E}[|X_n|] < \infty,$$

then $X_n \to X$ with probability 1 for some random variable X, and

$$\mathbb{E}[|X|] \leq K.$$

Idea: upcrossings measure oscillations; if a sequence converges, it cannot keep crossing the same interval infinitely many times.

Proof Sketch via Upcrossings

Given $\{X_n\}$ is a submartingale, assume

$$K := \sup_{n \geq 1} \mathbb{E}[|X_n|] < \infty. \tag{1}$$

Fix an interval $[\alpha, \beta]$ with $\alpha < \beta$, and define

$$U_n := \text{number of upcrossings of } [\alpha, \beta] \text{ by } X_1, \ldots, X_n.$$

From the upcrossing bound,

$$\mathbb{E}[U_n] \leq \frac{\mathbb{E}[|X_n|] + |\alpha|}{\beta - \alpha}. \tag{2}$$

From (1) and (2),

$$\mathbb{E}[U_n] \leq \frac{K + |\alpha|}{\beta - \alpha}. \tag{3}$$

Finishing the Upcrossing Argument

The sequence $\{U_n\}_{n\geq 1}$ is nonnegative and nondecreasing (it counts the oscillation intensity):

$$0 \leq U_1(\omega) \leq U_2(\omega) \leq U_3(\omega) \leq \cdots \qquad \forall\, \omega \in \Omega.$$

Define

$$U(\omega) := \lim_{n\to\infty} U_n(\omega) = \sup_{n\geq 1} U_n(\omega).$$

By the Monotone Convergence Theorem,

$$\mathbb{E}[U] = \lim_{n\to\infty} \mathbb{E}[U_n]. \tag{4}$$

Using (3),

$$\mathbb{E}[U_n] \leq \frac{K+|\alpha|}{\beta-\alpha} \quad \Longrightarrow \quad \mathbb{E}[U] \leq \frac{K+|\alpha|}{\beta-\alpha}. \tag{5}$$

Limsup, Liminf, and Oscillation

Now let's define

$$X^* := \limsup_{m\to\infty} X_m, \qquad X_* := \liminf_{m\to\infty} X_m.$$

- If $X_* < X^*$, then the sequence oscillates.
- If $X_* = X^*$, then it converges.

If

$$X_*(\omega) < \alpha < \beta < X^*(\omega),$$

then there are infinitely many upcrossings.

Consider $X_*(\omega) < \alpha$. By definition of $\liminf$,

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } X_k > X_* - \varepsilon\ \ \forall k \geq N_\varepsilon.$$

Consider $X^*(\omega) > \beta$. By definition of $\limsup$,

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } X_k < X^* + \varepsilon\ \ \forall k \geq N_\varepsilon$$

Density of Rationals and Convergence

This says that there are infinitely many indices with $X_k > \beta$. No matter how far we go, the sequence will go above β.

Now, we have shown that the number of upcrossings is finite (from the upcrossing bound), hence

$$\mathbb{P}\big(X_* < \alpha < \beta < X^*\big) = 0.$$

From the density of rationals: if for some real numbers we have $X_*(\omega) < X^*(\omega)$, then there exist rational numbers α, β such that

$$X_*(\omega) < \alpha < \beta < X^*(\omega).$$

Hence

$$\{X_* < X^*\} = \bigcup_{\substack{\alpha,\beta\in\mathbb{Q} \\ \alpha<\beta}} \{X_* < \alpha < \beta < X^*\}.$$

Countable Union Argument

Since this is a countable union and each event has probability zero,

$$\mathbb{P}(X_* < X^*) \leq \sum_{\substack{\alpha,\beta\in\mathbb{Q} \\ \alpha<\beta}} \mathbb{P}(X_* < \alpha < \beta < X^*) = 0.$$

Therefore $\mathbb{P}(X_* < X^*) = 0$, i.e.

$$X_* = X^* \quad \text{a.s.}$$

Defining the Limit and Applying Fatou

Define

$$X := \lim_{n\to\infty} X_n.$$

Apply Fatou's lemma with $Y_n := |X_n|$:

$$\mathbb{E}[|X|] = \mathbb{E}\Big[\lim_{n\to\infty} |X_n|\Big] = \mathbb{E}\Big[\liminf_{n\to\infty} |X_n|\Big] \le \liminf_{n\to\infty} \mathbb{E}[|X_n|] \le K.$$

Summary of the Argument

$K < \infty \implies (U_n)$ is bounded $\implies U < \infty$ a.s. $\implies$ no infinite oscillations $\implies X_n \to X$.

Doob's Decomposition

Theorem (Doob's decomposition)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. If $\{X_n\}_{n\geq 0}$ is a submartingale with respect to $(\mathscr{F}_n)_{n\geq 0}$, then there exist unique processes $(M_n)_{n\geq 0}$ and $(A_n)_{n\geq 0}$ such that

$$X_n = M_n + A_n \qquad \forall n \geq 0,$$

where

- (M_n) is a martingale with respect to $(\mathscr{F}_n)$,
- (A_n) is a predictable and increasing process:

$$A_0 = 0, \qquad A_n \text{ is } \mathscr{F}_{n-1}\text{-measurable}, \qquad A_n - A_{n-1} \geq 0 \text{ a.s.}$$

Doob's Decomposition (Idea)

Doob's decomposition separates a submartingale into:

- a fair game component (M_n) with no predictable gain,
- a drift component (A_n) that represents the accumulated conditional expected increase.

Doob's Decomposition: Constructing A_n

For $k \geq 1$,

$$\mathbb{E}[X_k - X_{k-1} \mid \mathscr{F}_{k-1}] = \mathbb{E}[X_k \mid \mathscr{F}_{k-1}] - X_{k-1}.$$

Since (X_n) is a submartingale,

$$\mathbb{E}[X_k \mid \mathscr{F}_{k-1}] \geq X_{k-1} \quad \Longrightarrow \quad \mathbb{E}[X_k - X_{k-1} \mid \mathscr{F}_{k-1}] \geq 0.$$

Define

$$A_n := \sum_{k=1}^{n} \mathbb{E}[X_k - X_{k-1} \mid \mathscr{F}_{k-1}].$$

Then

$$A_n - A_{n-1} = \mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}] \geq 0.$$

Hence (A_n) is increasing, and A_n is $\mathscr{F}_{n-1}$-measurable, i.e., (A_n) is predictable.

Doob's Decomposition: Defining M_n

Now we need to show that (M_n) is adapted and integrable.

Since X_n is $\mathscr{F}_n$-measurable and A_n is $\mathscr{F}_{n-1}$-measurable (hence $\mathscr{F}_n$-measurable), define

$$M_n := X_n - A_n.$$

Then M_n is $\mathscr{F}_n$-measurable.

Doob's Decomposition: M_n is a Martingale

We need to show that (M_n) is a martingale. Taking conditional expectation in the definition $M_n = X_n - A_n$ gives

$$\mathbb{E}[M_n \mid \mathscr{F}_{n-1}] = \mathbb{E}[X_n - A_n \mid \mathscr{F}_{n-1}] = \mathbb{E}[X_n \mid \mathscr{F}_{n-1}] - \mathbb{E}[A_n \mid \mathscr{F}_{n-1}].$$

From the construction,

$$A_n = A_{n-1} + \mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}].$$

Since A_{n-1} is $\mathscr{F}_{n-1}$-measurable,

$$\mathbb{E}[A_n \mid \mathscr{F}_{n-1}] = A_{n-1} + \mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}].$$

Therefore,

$$\mathbb{E}[M_n \mid \mathscr{F}_{n-1}] = \mathbb{E}[X_n \mid \mathscr{F}_{n-1}] - A_{n-1} - \mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}] = X_{n-1} - A_{n-1} = M_{n-1}.$$

Doob's Decomposition: Uniqueness

Suppose we have two decompositions

$$X_n = M_n + A_n = M'_n + A'_n,$$

where $(M_n), (M'_n)$ are martingales and $(A_n), (A'_n)$ are predictable and increasing with $A_0 = A'_0 = 0$.
Subtracting consecutive terms,

$$X_n - X_{n-1} = (M'_n - M'_{n-1}) + (A'_n - A'_{n-1}).$$

Taking conditional expectation with respect to $\mathscr{F}_{n-1}$ yields

$$\mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}] = \mathbb{E}[M'_n - M'_{n-1} \mid \mathscr{F}_{n-1}] + \mathbb{E}[A'_n - A'_{n-1} \mid \mathscr{F}_{n-1}].$$

Doob's Decomposition: Uniqueness (cont.)

Since (M'_n) is a martingale, the first term is 0. Also, $A'_n - A'_{n-1}$ is $\mathscr{F}_{n-1}$-measurable, so

$$\mathbb{E}[A'_n - A'_{n-1} \mid \mathscr{F}_{n-1}] = A'_n - A'_{n-1}.$$

Hence

$$\mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}] = A'_n - A'_{n-1} \qquad \text{a.s.}$$

By the same argument using $X_n = M_n + A_n$, we also have

$$\mathbb{E}[X_n - X_{n-1} \mid \mathscr{F}_{n-1}] = A_n - A_{n-1} \qquad \text{a.s.}$$

Therefore

$$A_n - A_{n-1} = A'_n - A'_{n-1} \qquad \text{a.s. for all } n \geq 1.$$

Doob's Decomposition: Uniqueness (cont.)

Define $D_n := A_n - A'_n$. Then

$$D_n - D_{n-1} = (A_n - A_{n-1}) - (A'_n - A'_{n-1}) = 0 \qquad \text{a.s.}$$

so $D_n = D_0$ a.s. for all n. Since $D_0 = A_0 - A'_0 = 0$, we get $A_n = A'_n$ a.s. Consequently $M_n = X_n - A_n = X_n - A'_n = M'_n$ a.s.