AIFES
AI for Finance, Economies and Society

Random walk

॥ नवोन्मेषेण देशनिर्माणम् ॥
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Random walk)

Let $\{\xi_n\}_{n\geq 1}$ be i.i.d. real-valued random variables.
Define

$$S_n := \sum_{k=1}^{n} \xi_k, \qquad S_0 := 0.$$

Note

S_n is adapted to the natural filtration $\mathscr{F}_n := \sigma(\xi_1, \ldots, \xi_n)$.

Symmetric random walk

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Symmetric random walk)

Let $\{X_k\}_{k \geq 1}$ be i.i.d. random variables such that, for each k,

$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}.$$

Define

$$S_n := \sum_{k=1}^{n} X_k, \qquad n \geq 1.$$

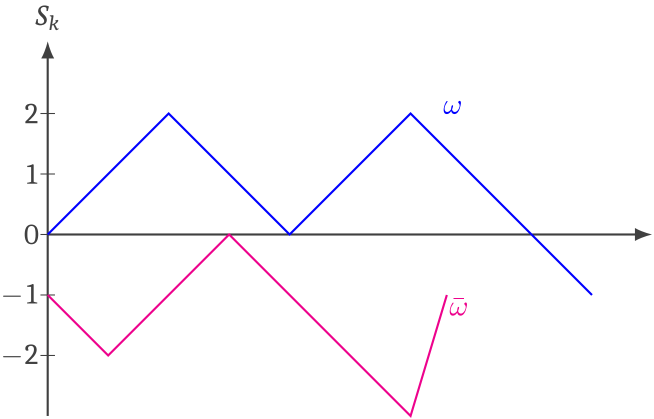
One scenario (coin-toss view)

Identify $H \leftrightarrow +1$ and $T \leftrightarrow -1$.

AIFES
AI for Finance, Economies and Society

Symmetric random walk: sample paths

॥ नवोन्मेषण देशनिर्माणम् ॥
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad



Properties of the symmetric random walk

Properties:

1. **Independent increments.** Fix indices $0 = k_0 < k_1 < k_2 < \cdots < k_n$ and define

$$\Delta_{k_i} := S_{k_i} - S_{k_{i-1}} = \sum_{j=1}^{k_i} X_j - \sum_{j=1}^{k_{i-1}} X_j = \sum_{j=k_{i-1}+1}^{k_i} X_j.$$

   Hence $\Delta_{k_1}, \Delta_{k_2}, \ldots, \Delta_{k_n}$ are independent random variables.

2. $\mathbb{E}[X_j] = 0 \;\; \forall j.$
3. $\mathbb{E}[S_n] = 0.$
4. $\mathrm{Var}(X_j) = 1.$
5. $\mathrm{Var}(S_n) = n$ (grows linearly).

More properties

Additional properties:

6. $\mathbb{E}[\Delta_{k_i}] = 0.$
7. $\mathrm{Var}(\Delta_{k_i}) = 1.$

Symmetric random walk is a martingale

AIFES
AI for Finance, Economies and Society

॥ नवोन्मेषण देशनिर्माणम् ॥
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Statement

The symmetric random walk $(S_k)_{k\geq 0}$ is a martingale with respect to its natural filtration $(\mathscr{F}_k)$, i.e.

$$\mathbb{E}[S_\ell \mid \mathscr{F}_k] = S_k, \qquad \ell > k.$$

Proof sketch:

1. S_k is $\mathscr{F}_k$-measurable.
2. (S_k) is adapted to $(\mathscr{F}_k)$.
3. $\mathbb{E}[|S_k|] < \infty$ for all k.

Now,

$$\mathbb{E}[S_\ell \mid \mathscr{F}_k] = \mathbb{E}[S_\ell - S_k \mid \mathscr{F}_k] + \mathbb{E}[S_k \mid \mathscr{F}_k] = \mathbb{E}[S_\ell - S_k] + S_k = S_k,$$

since $S_\ell - S_k = \sum_{j=k+1}^{\ell} X_j$ is independent of $\mathscr{F}_k$ and has mean 0.

Therefore the symmetric random walk has no drift: on average it does not move up or

Quadratic variation of the symmetric random walk

Quadratic variation is a measure of how strong the randomness is per unit time.

It quantifies the intensity of randomness in the path (not the randomness itself), i.e., a measure of randomness present in the path.

The quadratic variation up to time k is defined by

$$[S,S]_k := \sum_{j=1}^{k}(S_j - S_{j-1})^2 = \sum_{j=1}^{k}\left(\sum_{i=1}^{j}X_i - \sum_{i=1}^{j-1}X_i\right)^2 = \sum_{j=1}^{k}X_j^2 = k.$$

Here the computation is pathwise.

AIFES
AI for Finance, Economies and Society

Scaled random walk

Indian Institute of Technology Hyderabad

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $(S_n)_{n\geq 0}$ be a symmetric random walk.

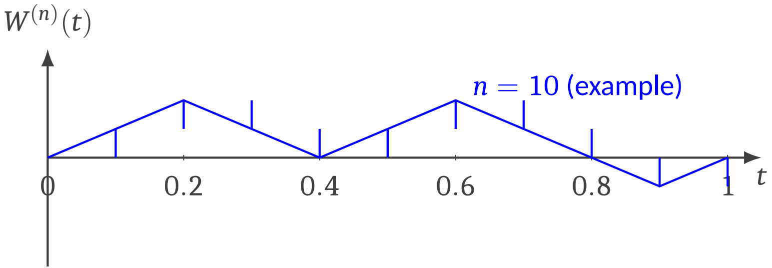
Definition (Scaled random walk)

For each $n \in \mathbb{N}$, define the process

$$W^{(n)}(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}, \qquad t \in [0, 1].$$

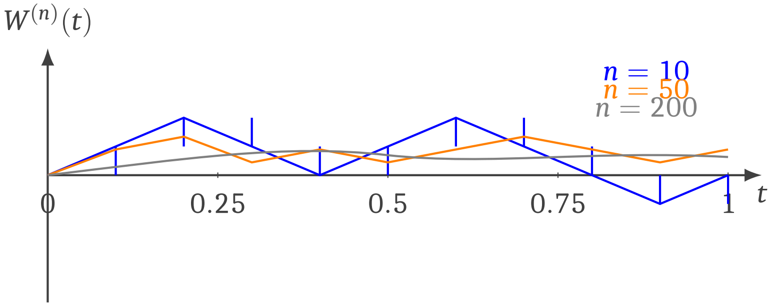
The idea is to speed up time, shrink space, and make the variance finite. The $1/\sqrt{n}$ scaling is taken from the CLT normalization and helps the random-walk increments converge to Gaussian increments.

Scaled random walk: sample path



Scaled random walk: comparison as n increases

For a fixed time horizon $t \in [0, 1]$, the step size in time is $1/n$ and the vertical scale is $1/\sqrt{n}$. As n increases, the path has more (smaller) jumps and looks more like a continuous curve.



Scaled random walk: basic properties

Properties:

1. **Independent increments.** For $t > s$,

$$W^{(n)}(t) - W^{(n)}(s) = \frac{1}{\sqrt{n}}\left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor}\right),$$

   and the increments inherit independence from (S_k).

2. **Mean.**

$$\mathbb{E}[W^{(n)}(t)] = \mathbb{E}\left[\frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}\right] = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \mathbb{E}[X_j] = 0.$$

3. **Variance.**

$$\mathrm{Var}(W^{(n)}(t)) = \frac{1}{n}\mathrm{Var}(S_{\lfloor nt \rfloor}) = \frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor}\mathrm{Var}(X_j) = \frac{\lfloor nt \rfloor}{n} \approx t.$$

Scaled random walk: increments

Let $t > s$.

4. Mean of increments

$$\mathbb{E}\left[W^{(n)}(t) - W^{(n)}(s)\right] = \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}\left[\frac{X_i}{\sqrt{n}}\right] - \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{E}\left[\frac{X_i}{\sqrt{n}}\right] = 0.$$

5. Variance of increments

$$\begin{aligned}
\mathrm{Var}\left(W^{(n)}(t) - W^{(n)}(s)\right) &= \mathrm{Var}\left(\frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor} - \frac{1}{\sqrt{n}}S_{\lfloor ns \rfloor}\right) = \frac{1}{n}\mathrm{Var}\left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor}\right) \\
&= \frac{1}{n}\mathrm{Var}\left(\sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} X_i\right) = \frac{1}{n}\sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor}\mathrm{Var}(X_i) \\
&= \frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} \approx t - s.
\end{aligned}$$

Filtration for the scaled random walk

The filtration associated with the scaled random walk is

$$\mathscr{F}_t^{(n)} := \sigma\left(X_1, X_2, \ldots, X_{\lfloor nt \rfloor}\right).$$

Scaled random walk is also a martingale

Statement

The scaled random walk $(W^{(n)}(t))_{t\in[0,1]}$ is a martingale with respect to $(\mathscr{F}_t^{(n)})$:

$$\mathbb{E}\left[W^{(n)}(t) \mid \mathscr{F}_s^{(n)}\right] = W^{(n)}(s), \qquad 0 \le s < t \le 1.$$

Proof sketch:

1. $W^{(n)}(t)$ is $\mathscr{F}_t^{(n)}$-measurable.
2. $\mathbb{E}[|W^{(n)}(t)|] < \infty$.

AIFES
AI for Finance, Economies and Society

Scaled random walk is also a martingale (cont.)

|| नवोन्मेषेण देशनिर्माणम् ||
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Proof sketch (cont.):

3. Using $W^{(n)}(t) = \big(W^{(n)}(t) - W^{(n)}(s)\big) + W^{(n)}(s)$,

$$\mathbb{E}\big[W^{(n)}(t) \mid \mathscr{F}_s^{(n)}\big] = \mathbb{E}\big[W^{(n)}(t) - W^{(n)}(s) \mid \mathscr{F}_s^{(n)}\big] + W^{(n)}(s) = 0 + W^{(n)}(s),$$

   since $W^{(n)}(t) - W^{(n)}(s)$ depends only on $X_{\lfloor ns \rfloor + 1}, \ldots, X_{\lfloor nt \rfloor}$ and is independent of $\mathscr{F}_s^{(n)}$ with mean 0.

Therefore, it is a martingale.

AIFES
AI for Finance, Economies and Society

Quadratic variation of the scaled random walk

॥ नवोन्मेषण देशनिर्माणम् ॥
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Fix n and define, for $t \in [0,1]$,

$$[W^{(n)}, W^{(n)}](t) := \sum_{j=1}^{\lfloor nt \rfloor} \left(W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right)\right)^2.$$

Using $W^{(n)}(j/n) = \frac{1}{\sqrt{n}} S_j$, we get

$$[W^{(n)}, W^{(n)}](t) = \sum_{j=1}^{\lfloor nt \rfloor} \left(\frac{1}{\sqrt{n}} S_j - \frac{1}{\sqrt{n}} S_{j-1} \right)^2 = \sum_{j=1}^{\lfloor nt \rfloor} \frac{1}{n} (S_j - S_{j-1})^2 = \sum_{j=1}^{\lfloor nt \rfloor} \frac{1}{n} X_j^2 = \frac{\lfloor nt \rfloor}{n} \approx t.$$

As $n \to \infty$,

$$[W^{(n)}, W^{(n)}](t) \to t \qquad \text{almost surely.}$$

Functional Central Limit Theorem

Functional CLT / Donsker's theorem / invariance principle

The classical CLT asserts that

$$W^{(n)}(1) \xrightarrow{d} W(1),$$

where $W(1) \sim \mathcal{N}(0, 1)$.

Donsker's theorem extends this convergence to the whole process $W^{(n)} := \left(W^{(n)}(t)\right)_{t\in[0,1]}$:

$$W^{(n)}(t) \xrightarrow{d} W(t),$$

where $(W(t))_{t\in[0,1]}$ is a standard Brownian motion (Wiener process).

Brownian motion

Definition (Brownian motion / Wiener process)

A process $\{W_t : t \geq 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a **Brownian motion** if:

(i) $\mathbb{P}(W_0 = 0) = 1$.

(ii) **Independent increments:** if $0 = t_0 \leq t_1 \leq \cdots \leq t_k$, then for any Borel sets $H_1, \ldots, H_k$,

$$\mathbb{P}\Big(W_{t_i} - W_{t_{i-1}} \in H_i \text{ for } i = 1, \ldots, k\Big) = \prod_{i=1}^{k} \mathbb{P}(W_{t_i} - W_{t_{i-1}} \in H_i).$$

(iii) **Gaussian increments:** for $0 \leq s < t$, $W_t - W_s \sim \mathcal{N}(0, t - s)$.

(iv) **Continuous paths:** for almost every ω, the map $t \mapsto W(t, \omega)$ is continuous.

Brownian motion: notes

Gaussian increment distribution: for $0 \le s < t$ and a Borel set $H \subset \mathbb{R}$,

$$\mathbb{P}(W_t - W_s \in H) = \frac{1}{\sqrt{2\pi(t-s)}} \int_H \exp\left(-\frac{x^2}{2(t-s)}\right) dx.$$

Note: if $0 < t_1 < \cdots < t_k$, then the joint density of $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}})$ is the product of the corresponding normal densities.

Filtration associated with Brownian motion

Note:

1. Brownian motion is continuous.
2. Brownian motion is nowhere differentiable.

Filtration:

1. For $0 \leq s \leq t$, $\mathscr{F}_s \subseteq \mathscr{F}_t$.
2. $W(t)$ must be adapted to the filtration.

Brownian motion is a martingale

Let $\mathscr{F}_t := \sigma(W_s : s \leq t)$. Then for any $s < t$,

$$\mathbb{E}[W_t \mid \mathscr{F}_s] = W_s.$$

Proof sketch:

$$\mathbb{E}[W_t \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s + W_s \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s \mid \mathscr{F}_s] + \mathbb{E}[W_s \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s] + W_s = W_s,$$

since $W_t - W_s$ is independent of $\mathscr{F}_s$ and has mean 0.

AIFES
AI for Finance, Economies and Society

Stationary increments

|| नवोन्मेषेण देशनिर्माणम् ||
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Statement

The increments of Brownian motion are stationary; for $0 \leq s < t$,

$$W_t - W_s \stackrel{d}{=} W_{t-s} - W_0.$$

Proof sketch: since $W_0 = 0$ and $W_t - W_s \sim \mathcal{N}(0, t-s)$, we have

$$W_{t-s} - W_0 = W_{t-s} \sim \mathcal{N}(0, t-s),$$

hence $W_t - W_s \stackrel{d}{=} W_{t-s} - W_0$.