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# Brownian motion

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## Definition (Brownian motion / Wiener process)

A process $\{W_t : t \geq 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a **Brownian motion** if:

(i) $\mathbb{P}(W_0 = 0) = 1.$

(ii) **Independent increments:** if $0 = t_0 \leq t_1 \leq \cdots \leq t_k$, then for any Borel sets $H_1, \ldots, H_k$,

$$\mathbb{P}\Big(W_{t_i} - W_{t_{i-1}} \in H_i \text{ for } i = 1, \ldots, k\Big) = \prod_{i=1}^{k} \mathbb{P}(W_{t_i} - W_{t_{i-1}} \in H_i).$$

(iii) **Gaussian increments:** for $0 \leq s < t$, $W_t - W_s \sim \mathcal{N}(0, t - s)$.

(iv) **Continuous paths:** for almost every $\omega$, the map $t \mapsto W(t, \omega)$ is continuous.

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## Brownian motion: notes

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**Gaussian increment distribution:** for $0 \leq s < t$ and a Borel set $H \subset \mathbb{R}$,

$$\mathbb{P}(W_t - W_s \in H) = \frac{1}{\sqrt{2\pi(t-s)}} \int_H \exp\left(-\frac{x^2}{2(t-s)}\right) dx.$$

**Note:** if $0 < t_1 < \cdots < t_k$, then the joint density of $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}})$ is the product of the corresponding normal densities.

# Filtration associated with Brownian motion

**Note:**

1. Brownian motion is continuous.
2. Brownian motion is nowhere differentiable.

**Filtration:**

1. For $0 \leq s \leq t$, $\mathscr{F}_s \subseteq \mathscr{F}_t$.
2. $W(t)$ must be adapted to the filtration.

# Brownian motion is a martingale

Let $\mathscr{F}_t := \sigma(W_s : s \leq t)$. Then for any $s < t$,

$$\mathbb{E}[W_t \mid \mathscr{F}_s] = W_s.$$

**Proof sketch:**

$$\mathbb{E}[W_t \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s + W_s \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s \mid \mathscr{F}_s] + \mathbb{E}[W_s \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s] + W_s = W_s,$$

since $W_t - W_s$ is independent of $\mathscr{F}_s$ and has mean 0.

# Stationary increments

**Statement**

The increments of Brownian motion are stationary; for $0 \leq s < t$,

$$W_t - W_s \stackrel{d}{=} W_{t-s} - W_0.$$

**Proof sketch:** since $W_0 = 0$ and $W_t - W_s \sim \mathcal{N}(0, t-s)$, we have

$$W_{t-s} - W_0 = W_{t-s} \sim \mathcal{N}(0, t-s),$$

hence $W_t - W_s \stackrel{d}{=} W_{t-s} - W_0$.

## Expectation and variance

Consider $0 \leq s \leq t$ and $W_t = (W_t - W_s) + W_s$.

$$W_s W_t = W_s(W_t - W_s) + W_s^2$$

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2]$$

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s]\,\mathbb{E}[W_t - W_s] + \mathbb{E}[W_s^2]$$

We know that

$$\mathbb{E}[W_s] = 0$$

$$\mathbb{E}[W_t - W_s] = 0$$

$$\therefore\ \mathbb{E}[W_s]\,\mathbb{E}[W_t - W_s] = 0$$

If $t \leq s$ then hen

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_t^2] = \text{var}(W_t) = t$$

$$\therefore\ \mathbb{E}[W_s W_t] = \min(s, t) = \text{cov}(W_s, W_t)$$

# Markov Property

Consider $\{W_t : t \geq 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is Brownian motion.

for $t_0 \geq 0$

$$\mathbb{P}(W_{t+t_0} \mid \mathscr{F}_{t_0}) = \mathbb{P}(W_{t+t_0} \mid W_{t_0}) \, .$$

To know $W_{t+t_0}$, all we need the current $W_{t_0}$. The entire history adds no information.

$$W_t' := W_{t_0+t} - W_{t_0}$$

$$W_{t_0+t} = W_{t_0} + W_t'$$

$$W_t' \sim \mathcal{N}(0, t)$$

# Markov Property (cont.)

1. $\left(W'_{t_1}, \ldots, W'_{t_k}\right)$ is independent of $\mathscr{F}_{t_0}$:

$$\mathscr{F}_{t_0} = \sigma\left(W_s : s \leq t_0\right)$$

consider

$$0 \leq s_1 \leq \cdots \leq s_j \leq t_0 \qquad \text{and} \qquad 0 \leq t_1 \leq \cdots \leq t_k$$

and $u_i = t_0 + t_i$.

Now

$$\left(W'_{t_1}, W'_{t_2} - W'_{t_1}, \ldots, W'_{t_k} - W'_{t_{k-1}}\right) = \left(W_{u_1} - W_{t_0}, \ldots, W_{u_k} - W_{u_{k-1}}\right). \tag{1}$$

# Finite Dimensional Distributions

## Finite Dimensional Distributions

Consider $(X_t)_{t \in T}$ $\quad T = [0, 1] \quad$ (or $\mathbb{R}$)

i.e, $\quad X_t(\omega)$ or $X(t, \omega), \ \omega \in \Omega$ and $t \in T$

fix an outcome $\omega$,

$$X(\cdot, \omega) : T \longrightarrow \mathbb{R}$$

For an outcome, it is a function in space $\mathbb{R}^T$.

For all outcomes, functions of functions

# Joint probability law of random variables

## Joint probability law of random variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Given $X_1, X_2, \ldots, X_n$ defined with respect to $\mathscr{F}$, their joint probability law is the probability measure

$$P_{X_1,\ldots,X_n} : \mathcal{B}(\mathbb{R}^n) \to [0, 1]$$

defined as

$$P_{X_1,\ldots,X_n}(B) = \mathbb{P}\Big(\{\omega \in \Omega : (X_1(\omega), \ldots, X_n(\omega)) \in B\}\Big), \qquad B \in \mathcal{B}(\mathbb{R}^n).$$

# Finite-dimensional distributions (continued)

To define stochastic process usually describe in terms of distributions.

Consider

1. $k$-tuple $(t_1, t_2, \ldots, t_k)$ of distinct elements of $T$.
2. random vector $(X_{t_1}, \ldots, X_{t_k})$ has over $\mathbb{R}^k$ some measure $\mu_{t_1,t_2,\ldots,t_k}$.

$$\mu_{t_1,t_2,\ldots,t_k}(B) = \mathbb{P}\Big(\{\omega \in \Omega : (X_{t_1}(\omega), \ldots, X_{t_k}(\omega)) \in B\}\Big), \qquad B \in \mathcal{B}(\mathbb{R}^k).$$

$$= P_{X_{t_1},\ldots,X_{t_k}}(B).$$

The collection of probability measures $\mu_{t_1,t_2,\ldots,t_k}$ (or $P_{X_{t_1},\ldots,X_{t_k}}$) are the Finite dimensional distributions of the stochastic process $[X_t : t \in T]$.

# FDD implies 2 properties

The FDD implies 2 properties

1. Suppose $B = B_1 \times B_2 \times \cdots \times B_k \quad (B_j \in \mathcal{B}(\mathbb{R}^1))$
consider a permutation $\pi$ of $(1, 2, \ldots, k)$

$$\mu_{t_1,\ldots,t_k}\left(B_1 \times B_2 \times \cdots \times B_k\right) = \mu_{t_{\pi(1)},\ldots,t_{\pi(k)}}\left(B_{\pi(1)} \times B_{\pi(2)} \times \cdots \times B_{\pi(k)}\right).$$

2. The second consistency condition is

$$\mu_{t_1,\ldots,t_{k-1}}\left(B_1 \times B_2 \times \cdots \times B_{k-1}\right) = \mu_{t_1,\ldots,t_{k-1},t_k}\left(B_1 \times B_2 \times \cdots \times B_{k-1} \times \mathbb{R}^1\right).$$

# FDD consistency: explanation

**Explanation:**

$$\mu_{t_1,\dots,t_{k-1},t_k}\left(B_1 \times B_2 \times \cdots \times B_{k-1} \times \mathbb{R}^1\right)$$

= $\mathrm{P}\left(X_{t_1} \in B_1, \dots, X_{t_{k-1}} \in B_{k-1}, X_{t_k} \in \mathbb{R}\right)$.

We know that

$$\mathbb{P}(Y \in B) = \mathbb{P}\left(\{\omega \in \Omega : Y(\omega) \in B\}\right).$$

$$\therefore\ \mu_{t_1,\dots,t_k}\left(B_1 \times B_2 \times \cdots \times B_{k-1} \times \mathbb{R}^1\right) = \mathbb{P}\left(\bigcap_{i=1}^{k-1}\{\omega \in \Omega : X_{t_i}(\omega) \in B_i\}\right) \cap \{\omega \in \Omega : X_{t_k}(\omega) \in \mathbb{R}\}.$$

$$= \mathbb{P}\left(X_{t_1} \in B_1, X_{t_2} \in B_2, \dots, X_{t_{k-1}} \in B_{k-1}\right) = \mu_{t_1,\dots,t_{k-1}}\left(B_1 \times B_2 \times \cdots \times B_{k-1}\right).$$

# Kolmogorov's Existence Theorem

**Kolmogorov's Existence Theorem**

If $\mu_{t_1,\ldots,t_k}$ are a system of distributions satisfying the consistency conditions, then there exists on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ a stochastic process $[X_t : t \in T]$ having the $\mu_{t_1,\ldots,t_k}$ as its finite dimensional distributions.

# Existence of Brownian motion

**Existence of Brownian motion:**

Let $\mu_{t_1,\ldots,t_k}$ be the distribution on $(S_1, \ldots, S_k)$

$$\text{where} \quad S_i = \sum_{j=1}^{i} X_j \quad \text{and} \quad X_i \sim \mathcal{N}(0, t_i - t_{i-1}) \ \forall t \geq 2, \qquad X_1 \sim \mathcal{N}(0, t_1).$$

# Permutation consistency

**Permutation consistency:**

$$\mu_{t_1,\dots,t_k} = \mathbb{P}(S_1 \in B_1, S_2 \in B_2, \dots, S_k \in B_k), \qquad B_i \in \mathcal{B}(\mathbb{R}).$$

$$= \mathbb{P}\Big(\bigcap_{i=1}^{k} \{\omega \in \Omega : S_i \in B_i\}\Big).$$

Consider a permutation $\pi$

$$\mu_{t_{\pi(1)},\dots,t_{\pi(k)}} = \mathbb{P}\big(S_{\pi(1)} \in B_{\pi(1)}, \dots, S_{\pi(k)} \in B_{\pi(k)}\big).$$

$$= \mathbb{P}\Big(\bigcap^{k} \{\omega \in \Omega : S_{\pi(i)} \in B_{\pi(i)}\}\Big).$$

# Permutation Consistency Cont.

Since $\pi$ is a permutation, as $j$ runs 1 to $k$, $i = \pi(j)$ also runs from 1 **to** $k$.

$$\therefore \ \mu_{t_1,\ldots,t_k} = \mu_{t_{\pi(1)},\ldots,t_{\pi(k)}}.$$

# Marginalization consistency

**Marginalization consistency:** Consider a function $g$

$$g(x_1, x_2, \ldots, x_k) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k).$$

$$\therefore\ g(S_1, S_2, \ldots, S_k) = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k).$$

Without loss of generality

$\mu_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_k}$ has distribution for $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k$

So

$$\mu_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_k} = \mathbb{P}\big(S_1 \in B_1, \ldots, S_{i-1} \in B_{i-1}, S_{i+1} \in B_{i+1}, \ldots, S_k \in B_k\big).$$

# Marginalization consistency

$$\mu_{t_1,\dots,t_k} = \mathbb{P}\big(S_1 \in B_1, \dots, S_i \in \mathbb{R}, \dots, S_k \in B_k\big).$$

$$\therefore \ \mu_{t_1,\dots,t_k} = \mu_{t_1,\dots,t_{i-1},t_{i+1},\dots,t_k}.$$

$\therefore$ A process $[W_t : t > 0]$ considering $W_t = 0$ at $t = 0$ exists a some $(\Omega, \mathscr{F}, \mathbb{P})$.

# Markov property

Consider $\{W_t : t \geq 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is Brownian motion.
For $t_0 \geq 0$,

$$\mathbb{P}\left(W_{t+t_0} \mid \mathscr{F}_{t_0}\right) = \mathbb{P}\left(W_{t+t_0} \mid W_{t_0}\right).$$

To know $W_{t+t_0}$, all we need the current $W_{t_0}$. The entire history adds no information.

Define

$$W'_t := W_{t_0+t} - W_{t_0}.$$

$$W'_t \sim \mathcal{N}(0, t).$$

1. $(W'_{t_1}, \ldots, W'_{t_k})$ is independent of $\mathscr{F}_{t_0}$:

$$\mathscr{F}_{t_0} = \sigma(W_s : s \leq t_0).$$

Consider $0 \leq s_1 \leq \cdots \leq s_j \leq t_0$ and $0 \leq t_1 \leq \cdots \leq t_k$, and $u_i = t_0 + t_i$.

# Markov property (cont.)

and also

$$W_{s_1},\ W_{s_2}-W_{s_1},\ W_{s_3}-W_{s_2},\ \ldots,\ W_{s_j}-W_{s_{j-1}}. \tag{2}$$

From (??), the intervals

$$[s_1,s_2],\ldots,[s_{j-1},s_j] \qquad \text{end before } t_0,$$

and

$$[t_0,u_1],\ldots,[u_{k-1},u_k] \qquad \text{start at or after } t_0.$$

So, the above are all disjoint intervals; hence, from the above properties, the increments over disjoint intervals are independent, i.e.,

$$(W'_{t_1},\ldots,W'_{t_k}) \perp (W_{s_1},\ldots,W_{s_j}),$$

and therefore

$$(W'_{t_1},\ldots,W'_{t_k}) \perp \mathscr{F}_{t_0}.$$

# Markov property (cont.)

Consider, for $B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}(W_{t_0+t} \in B \mid \mathscr{F}_{t_0}) = \mathbb{E}\left[\mathbf{1}_{\{W_{t_0+t} \in B\}} \mid \mathscr{F}_{t_0}\right] = \mathbb{E}\left[\mathbf{1}_{\{W_{t_0}+W'_t \in B\}} \mid \mathscr{F}_{t_0}\right].$$

Here $W_{t_0}$ is $\mathscr{F}_{t_0}$-measurable and $W'_t$ is independent of $\mathscr{F}_{t_0}$. Hence,

$$\mathbb{E}\left[\mathbf{1}_{\{W_{t_0}+W'_t \in B\}} \mid \mathscr{F}_{t_0}\right] = \varphi(W_{t_0}),$$

for some (Borel) function $\varphi$.

## Markov property (cont.)

Now consider

$$\mathbb{E}\left[\mathbf{1}_{\{W_{t_0+t}\in B\}} \mid \sigma(W_{t_0})\right] = \mathbb{E}\left[\mathbf{1}_{\{W_{t_0}+W_t'\in B\}} \mid \sigma(W_{t_0})\right].$$

Since $\sigma(W_{t_0}) \subset \mathscr{F}_{t_0}$ and $W_t'$ is independent of $\sigma(W_{t_0})$, we again get $\varphi(W_{t_0})$.

Therefore,

$$\mathbb{E}\left[\mathbf{1}_{\{W_{t_0}+W_t'\in B\}} \mid \sigma(W_{t_0})\right] = \varphi(W_{t_0}).$$

$$\therefore\ \mathbb{P}\left(W_{t_0} + W_t' \in B \mid \mathscr{F}_{t_0}\right) = \mathbb{P}\left(W_{t_0} + W_t' \in B \mid \sigma(W_{t_0})\right),$$

and hence

$$\therefore\ \mathbb{P}(W_{t_0+t} \in B \mid \mathscr{F}_{t_0}) = \mathbb{P}(W_{t_0+t} \in B \mid \sigma(W_{t_0}))\,.$$

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## Markov property (cont.)



**The other way:** consider

$$\mathbb{P}(W_{t_0+t} \in B \mid \mathscr{F}_{t_0}) = \mathbb{P}(W_{t_0} + (W_{t_0+t} - W_{t_0}) \in B \mid \mathscr{F}_{t_0}) \,.$$

Since $\sigma(W_{t_0}) \subset \mathscr{F}_{t_0}$ and $(W_{t_0+t} - W_{t_0}) = W'_t$ is independent of $\mathscr{F}_{t_0}$ (hence independent of $\sigma(W_{t_0})$),

$$\therefore\; \mathbb{P}(W_{t_0+t} \in B \mid \mathscr{F}_{t_0}) = \mathbb{P}(W_{t_0+t} \in B \mid \sigma(W_{t_0})) \,.$$

# Strong Markov Property

Consider $\tau$ is a stopping time and also

$$W_t^*(\omega) = W_{\tau(\omega)+t}(\omega) - W_{\tau(\omega)}(\omega), \qquad t \geq 0.$$

Consider $\tau$ has countable range $V$, i.e., $V \subset [0, \infty)$.
Let $t_0$ be some general point of $V$.

$$\mathscr{F}_\tau = \{M \in \mathscr{F} : M \cap \{\omega : \tau(\omega) \leq t\} \in \mathscr{F}_t, \, \forall t \geq 0\}.$$

Consider

$$\{\omega : W_t^*(\omega) \in B\} = \bigcup_{t_0 \in V} \{\omega : W_{t_0+t}(\omega) - W_{t_0}(\omega) \in B, \; \tau(\omega) = t_0\}.$$

Now $A \in \mathscr{F}_\tau$.

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## Strong Markov Property (cont.)

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Now $A \in \mathscr{F}_\tau$.

$$\mathbb{P}\big(\{(W^*_{t_1}, \dots, W^*_{t_k}) \in B\} \cap A\big) = \sum_{t_0 \in V} \mathbb{P}\big(\{(W^*_{t_1}, \dots, W^*_{t_k}) \in B\} \cap A \cap \{\tau = t_0\}\big)$$

$$= \sum_{t_0 \in V} \mathbb{P}\big(\{(W^*_{t_1}, \dots, W^*_{t_k}) \in B\}\big)\ \mathbb{P}(A \cap \{\tau = t_0\})$$

$$= \mathbb{P}\big(\{(W^*_{t_1}, \dots, W^*_{t_k}) \in B\}\big) \sum_{t_0 \in V} \mathbb{P}(A \cap \{\tau = t_0\})$$

$$= \mathbb{P}\big(\{(W^*_{t_1}, \dots, W^*_{t_k}) \in B\}\big)\ \mathbb{P}(A).$$

## Strong Markov Property (cont.)

The above holds for $\tau$ having countable range. Now for any general $\tau$, consider

$$\tau_n = \begin{cases} k \cdot 2^{-n} & \text{if } (k-1)2^{-n} < \tau \leq k \cdot 2^{-n}, \quad k = 1, 2, \dots, \\ 0 & \tau = 0. \end{cases}$$

This is dyadic approximation which is used to approximate real numbers using dyadic rationals, where a number $x$ is approximated by finding an integer $k$ such that

$$\frac{k}{2^n} \leq x < \frac{k+1}{2^n}.$$

# Strong Markov Property (cont.)

At $\tau_n(\omega) = k \cdot 2^{-n}$ we know that

$$\tau(\omega) \le k \cdot 2^{-n}, \qquad \therefore\ \tau(\omega) \le \tau_n(\omega),$$

and

$$\tau(\omega) > (k-1)2^{-n} \quad \Longrightarrow \quad \tau(\omega) + 2^{-n} \ge k \cdot 2^{-n}.$$

Therefore

$$\therefore\ \tau(\omega) \le \tau_n(\omega) \le \tau(\omega) + 2^{-n}.$$

Hence, as $n \to \infty$,

$$\tau_n(\omega) \longrightarrow \tau(\omega).$$

# Strong Markov Property (cont.)

$\tau_n$ **is a stopping time:** if $k \cdot 2^{-n} \leq t < (k+1) \cdot 2^{-n}$ then

$$\{\tau_n \leq t\} = \{\tau \leq k \cdot 2^{-n}\} \in \mathscr{F}_{k \cdot 2^{-n}} \subset \mathscr{F}_t.$$

Suppose $A \in \mathscr{F}_\tau$ and $k \cdot 2^{-n} \leq t < (k+1) \cdot 2^{-n}$ then

$$A \cap \{\tau_n \leq t\} = A \cap \{\tau \leq k \cdot 2^{-n}\} \in \mathscr{F}_{k \cdot 2^{-n}} \subset \mathscr{F}_t.$$

Hence all such $A$'s lie in $\mathscr{F}_{\tau_n}$, i.e.,

$$A \in \mathscr{F}_\tau \quad \Longrightarrow \quad A \in \mathscr{F}_{\tau_n}.$$

# Strong Markov Property (cont.)

As $\tau_n(\omega) \to \tau(\omega)$ and Brownian paths are continuous,

$$W_t^{*(n)}(\omega) \longrightarrow W_t^*(\omega).$$

Therefore,

$$\mathbb{P}\Big(\{(W_{t_1}^{*(n)}, \ldots, W_{t_k}^{*(n)}) \in B\} \cap A\Big) = \mathbb{P}(\{(W_{t_1}^*, \ldots, W_{t_k}^*) \in B\} \cap A)$$

$$= \mathbb{P}(\{(W_{t_1}^*, \ldots, W_{t_k}^*) \in B\})\,\mathbb{P}(A).$$

$$\therefore\ (W_{t_1}^*, \ldots, W_{t_k}^*) \perp \mathscr{F}_\tau.$$

$$\therefore\ \mathbb{P}(W_{\tau+t} \in B \mid \mathscr{F}_\tau) = \mathbb{P}(W_{\tau+t} \in B \mid W_\tau)\,.$$

# Reflection principle

**Reflection principle:** $(W_t)_{t\geq 0}$ is a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\tau$ be the stopping time and define the reflected process $W''_t$ by

$$W''_t = \begin{cases} W_t, & t \leq \tau, \\ W_\tau - (W_t - W_\tau), & t \geq \tau, \end{cases}$$

$$= \begin{cases} W_t, & t \leq \tau, \\ 2W_\tau - W_t, & t \geq \tau. \end{cases}$$

$W''$ is Brownian motion.

The sample path for $\{W''_t : t \geq 0\}$ is the same as the sample path for $\{W_t : t \geq 0\}$ up to $\tau$. After that, it is reflected through the point $W_\tau$.

# Reflection principle

Consider $\tau$ has some countable range $V$.

Consider two times $s, t$ for $\{\tau = t_0\}$.

**Case 1:** $s \leq t \leq t_0$

$$W''_s = W_s, \qquad W''_t = W_t$$
$$W''_s \stackrel{d}{=} W_s, \qquad W''_t \stackrel{d}{=} W_t.$$

**Case 2:** $s \leq t_0 \leq t$

$$W''_s = W_s, \qquad W''_t = 2W_{t_0} - W_t.$$

Consider

$$W_t = W_{t_0} + (W_t - W_{t_0}).$$
$$W_t - W_{t_0} \sim \mathcal{N}(0, t - t_0)$$

# Reflection Principle

and Gaussian increments are symmetric, so

$$W_t \overset{d}{=} W_{t_0} - (W_t - W_{t_0}) = W_t''.$$

Hence

$$W_s \overset{d}{=} W_s'', \qquad W_t \overset{d}{=} W_t''.$$

**Case 3:** $t_0 \leq s \leq t$

$$W_s'' = 2W_{t_0} - W_s, \qquad W_t'' = 2W_{t_0} - W_t.$$

By the same argument in Case 2,

$$W_s \overset{d}{=} W_s'', \qquad W_t \overset{d}{=} W_t''.$$

$$\therefore\ (W_{t_1}, \ldots, W_{t_k}) \overset{d}{=} (W_{t_1}'', \ldots, W_{t_k}'').$$

# Reflection principle: distribution of the maximum

Now consider,

$$\tau = \inf\{s : W_s \geq x\},$$

the first time $W_s$ hits $x$ and consider

$$M_t = \sup_{s \leq t} W_s.$$

$M_t$ is the highest point that Brownian paths reaches before time $t$.

$$M_t \geq x \iff \exists s \in [0, t] \text{ such that } W_s \geq x.$$

The path reaches $x$ before time $t$ and $\tau$ is the first time path reaches level $x$.

If the path ever exceed $x$ then the path must hit $x$ before time $t$.

$$\mathbb{P}(M_t \geq x) = \mathbb{P}(\tau \leq t).$$

# Reflection principle: distribution of the maximum

$$\mathbb{P}(M_t \geq x) = \mathbb{P}(\tau \leq t,\ W_t \leq x) + \mathbb{P}(\tau \leq t,\ W_t \geq x)$$

$$= \mathbb{P}(\tau \leq t,\ W_t'' \geq x) + \mathbb{P}(\tau \leq t,\ W_t \geq x) \qquad \text{(by reflection principle)}$$

$$= \mathbb{P}(\tau \leq t,\ W_t \geq x) + \mathbb{P}(\tau \leq t,\ W_t \geq x)$$

since

$$\mathbb{P}(W_t \geq x) = \frac{1}{\sqrt{2\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-u^2/2}\, du.$$

$$\mathbb{P}(M_t \geq x) = \mathbb{P}(W_t \geq x) = \frac{2}{\sqrt{2\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-u^2/2}\, du.$$