Riemann Integral

Consider $f : [a, b] \to \mathbb{R}$, a bounded and continuous function, and Π_n is a partition of $[a, b]$ where

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

$$\|\Pi_n\| = \max_{1 \le i \le n} (t_i - t_{i-1})$$

f is said to be Riemann integrable if

$$\int_a^b f(t)\, dt = \lim_{\|\Pi_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1})$$

τ_i is the evaluation point in the interval $[t_{i-1}, t_i]$.

$\int_a^b f(t)\, dt$ exists as the limit exists as the mesh approaches zero

Riemann–Stieltjes Integral

Consider $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$.
f is bounded and g is a monotonically increasing function and Π_n is a partition of $[a, b]$ where

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

$$\|\Pi_n\| = \max_{1 \le i \le n} (t_i - t_{i-1}) \qquad (\|\Pi_n\| \text{ is called the mesh})$$

The function f with respect to g is said to be Riemann–Stieltjes integrable.

$$\int_a^b f(t)\, dg(t) = \lim_{\|\Pi_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)\big(g(t_i) - g(t_{i-1})\big)$$

τ_i is the evaluation point in the interval $[t_{i-1}, t_i]$.

Riemann–Stieltjes Integral (cont.)

$\int_a^b f(t)\, dg(t)$ exists as the limit exists as the mesh approaches zero

For $g(t) = t$, the Riemann–Stieltjes integral becomes the Riemann integral.

Note

Note:

1. A bounded function f is Riemann integrable if and only if it is continuous almost everywhere with respect to the Lebesgue measure.
i.e. the set of points where f is discontinuous has a Lebesgue measure 0.

2. A bounded function f is Riemann–Stieltjes integrable if and only if the set of points where f is discontinuous must have a measure of zero induced by g.

Mean Value Theorem

A function $f(t)$ is continuous on $[a,b]$ and differentiable on (a,b); there exists at least one number c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{for } a < c < b$$

$$\therefore\ f'(\tau_j) = \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} \qquad \text{for } t_j < \tau_j < t_{j+1}$$

Total Variation / First-Order Variation

Consider $f(t)$, a function defined on the interval,

$$f : [0, T] \to \mathbb{R}.$$

Π_n is a partition of $[0, T]$ where

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T$$

$$\|\Pi_n\| = \max_{1 \le i \le n} (t_i - t_{i-1})$$

The total variation is defined as

$$FV_T(f) = \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

Total Variation / First-Order Variation (cont.)

It is a measure of the total oscillation a function undergoes over the interval $[0, T]$.

If the function f has continuous derivative $f'(t)$ from mean value theorem,

$$FV_T(f) = \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

$$= \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} |f'(\tau_i)| (t_{i+1} - t_i)$$

$$= \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} |f'(\tau_i)| (t_{i+1} - t_i)$$

$$\boxed{FV_T(f) = \int_0^T |f'(t)| \, dt}$$

Quadratic Variation

Consider $f(t)$, a function defined on the interval,

$$f : [0, T] \to \mathbb{R}.$$

Π_n is a partition of $[0, T]$ where

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T$$

$$\|\Pi_n\| = \max_{1 \le i \le n} (t_i - t_{i-1})$$

The quadratic variation is defined as

$$[f, f](T) = \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} \big(f(t_{i+1}) - f(t_i)\big)^2$$

Quadratic Variation (continuous derivative)

Suppose the function f has a continuous derivative, then

$$\begin{aligned}
\sum_{i=0}^{n-1}\big(f(t_{i+1}) - f(t_i)\big)^2 &= \sum_{i=0}^{n-1}\left|f'(\tau_i)\right|^2 (t_{i+1} - t_i)^2 \\
&\leq \|\Pi_n\| \sum_{i=0}^{n-1}\left|f'(\tau_i)\right|^2 (t_{i+1} - t_i) \\
[f,f](T) &\leq \lim_{\|\Pi_n\|\to 0} \|\Pi_n\| \sum_{i=0}^{n-1}\left|f'(\tau_i)\right|^2 (t_{i+1} - t_i) \\
&= \lim_{\|\Pi_n\|\to 0} \|\Pi_n\| \lim_{\|\Pi_n\|\to 0} \sum_{i=0}^{n-1}\left|f'(\tau_i)\right|^2 (t_{i+1} - t_i)
\end{aligned}$$

Quadratic Variation (continuous derivative cont.)

$$[f,f](T)[f,f](T) = \lim_{\|\Pi\| \to 0} \|\Pi\| \int_0^T \left| f'(t) \right|^2 \, dt$$

If $f'(t)$ is continuous then $\int_0^T \left| f'(t) \right|^2 \, dt$ is finite

$$\therefore \; [f,f](T) = 0$$

Note

Note:
Not only continuously differentiable functions; there are also other classes of functions that are Riemann integrable with bounded variation and zero quadratic variation. Examples: continuous monotone functions, Lipschitz continuous functions, etc.

Result: If quadratic variation of a real function over an interval is positive, then the total variation of the function over the interval is infinite (not vice versa)
This we will see while discussing Total Variation of Brownian Motion

Quadratic Variation of Brownian Motion

Consider a probability space $(\Omega, \mathcal{F}, P)$, and $\{W_t\}_{t\geq 0}$ is a Brownian motion.
Π_n is a partition of $[0, T]$ where $t_0 = 0 < t_1 < \cdots < t_n = T$.

Definition (Sample Quadratic Variation)

The sample quadratic variation of brownian motion is given as

$$Q_{\Pi_n} = \sum_{i=0}^{n-1} \left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)^2 \qquad \omega \in \Omega$$

Q_{Π_n} are sequence of RV.

$$Q_{\Pi_n} \xrightarrow{L^2} T \qquad \text{and} \qquad Q_{\Pi_n} \xrightarrow{a.s} T$$

Expectation of Q_{Π_n}

Proof: (i) L^2 convergence:

$$\mathbb{E}[Q_{\Pi_n}] = \mathbb{E}\left[\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2\right]$$

$$= \sum_{i=0}^{n-1}\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^2\right]$$

$$= \sum_{i=0}^{n-1}\text{var}\left[(W_{t_{i+1}} - W_{t_i})\right] = \sum_{i=0}^{n-1}(t_{i+1} - t_i)$$

$$\therefore \boxed{\mathbb{E}[Q_{\Pi_n}] = T} (1)$$

Variance of Q_{Π_n}

$$
\begin{aligned}
\operatorname{var}(Q_{\Pi_n}) &= \operatorname{var}\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2\right) \\
&= \sum_{i=0}^{n-1} \operatorname{var}\left((W_{t_{i+1}} - W_{t_i})^2\right) \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^4\right] - \left(\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^2\right]\right)^2 \\
&= \sum_{i=0}^{n-1} 3\left(\operatorname{var}(W_{t_{i+1}} - W_{t_i})\right)^2 - \left(\operatorname{var}(W_{t_{i+1}} - W_{t_i})\right)^2 \quad = 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2(2)
\end{aligned}
$$

Bounding Variance of Q_{Π_n}

$$\mathbb{E}\left[(Q_{\Pi_n} - \mathbb{E}[Q_{\Pi_n}])^2\right] = \mathrm{var}(Q_{\Pi_n})$$

$$\begin{aligned}
\therefore\ \mathbb{E}\left[(Q_{\Pi_n} - T)^2\right] &= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \\
&= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)(t_{i+1} - t_i) \\
&\leq 2\sum_{i=0}^{n-1}\|\Pi_n\|(t_{i+1} - t_i) \\
&= 2\|\Pi_n\|\sum_{i=0}^{n-1}(t_{i+1} - t_i)
\end{aligned}$$

L^2 Convergence of Q_{Π_n}

$$\therefore \boxed{\mathbb{E}\left[(Q_{\Pi_n} - T)^2\right] \leq 2\|\Pi_n\|\, T} (3)$$

$$\therefore \mathbb{E}\left[Q_{\Pi_n}^2 - 2Q_{\Pi_n}T + T^2\right] \leq 2\|\Pi_n\|\, T,$$

$$\begin{aligned}
\mathbb{E}[Q_{\Pi_n}^2] &\leq 2\|\Pi_n\|\, T + 2\,\mathbb{E}[Q_{\Pi_n}]\, T - T^2 \\
&\leq 2\|\Pi_n\|\, T + T^2 \\
&\leq C \text{ (some constant)} < +\infty
\end{aligned}$$

$$\mathbb{E}[Q_{\Pi_n}^2] < +\infty \qquad \mathbb{E}[Q_{\Pi_n}] < +\infty$$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[(Q_{\Pi_n} - T)^2] &= \lim_{\|\Pi_n\|\to 0} \mathbb{E}[(Q_{\Pi_n} - T)^2] \\
&= 0 \qquad (\, n \to \infty, \text{ if } \|\Pi_n\| \to 0)
\end{aligned}$$

Quadratic Variation of Brownian Motion (cont.)

(ii) Almost sure convergence:

for $\varepsilon > 0$

$$\begin{aligned} P(|Q_{\Pi_n} - T| > \varepsilon) &= P\left((Q_{\Pi_n} - T)^2 > \varepsilon^2\right) \\ &\leq \frac{\mathbb{E}\left[(Q_{\Pi_n} - T)^2\right]}{\varepsilon^2} \qquad (\because \text{ Chebyshev inequality}) \\ &\leq \frac{2\|\Pi_n\| \, T}{\varepsilon^2} \end{aligned}$$

For $\sum_{n=1}^{\infty} P(|Q_{\Pi_n} - T| > \varepsilon) < +\infty$, need to show

$$\frac{2\|\Pi_n\| \, T}{\varepsilon^2} < \infty \qquad \text{i.e,} \quad \sum_{n=1}^{\infty} \|\Pi_n\| \text{ must be finite}$$

Quadratic Variation of Brownian Motion (cont.)

Consider dyadic approximation

$$\|\Pi_n\| = \frac{T}{2^n}$$

$$\sum_{n=1}^{\infty} \|\Pi_n\| = \sum_{n=1}^{\infty} \frac{T}{2^n}$$
$$< \infty$$

$$\therefore \sum_{n=1}^{\infty} P(|Q_{\Pi_n} - T| > \varepsilon) < \infty$$

$$\therefore \; Q_{\Pi_n} \xrightarrow{a.s} T \qquad (\because \text{ Borel Cantelli Lemma})$$

Total Variation of Brownian Motion

Consider a probability space $(\Omega, \mathcal{F}, P)$ and $\{W_t\}_{t\geq 0}$ is a Brownian motion.
Π_n is a partition of $[0, T]$ where

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T$$

$$\|\Pi_n\| = \max_{1\leq i\leq n} (t_i - t_{i-1})$$

Definition (Sample First-Order Variation)

The sample first order variation is given as

$$TV_{\Pi_n} = \sum_{i=0}^{n-1} \left| W_{t_{i+1}}(\omega) - W_{t_i}(\omega) \right|, \; \omega \in \Omega$$

Total Variation of Brownian motion (cont.)

and we know the sample quadratic variation is

$$
\begin{aligned}
Q_{\Pi_n} &= \sum_{i=0}^{n-1} \left(W_{t_{i+1}}(\omega) - W_{t_{i+1}}(\omega)\right)^2,\ \omega \in \Omega \\
&= \sum_{i=0}^{n-1} \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right) \\
&= \sum_{i=0}^{n-1} \left|W_{t_{i+1}} - W_{t_i}\right| \left|W_{t_{i+1}} - W_{t_i}\right| \\
&\leq \left(\max_{1 \leq i < n} \left(\left|W_{t_{i+1}} - W_{t_i}\right|\right)\right) \sum_{i=0}^{n-1} \left|W_{t_{i+1}} - W_{t_i}\right|
\end{aligned}
$$

Total Variation of Brownian motion (cont.)

as $\|\Pi_n\| \to 0$

$$Q_{\Pi_n} \to T$$

and

$$\max_{1 \le i < n} \left| W_{t_{i+1}} - W_{t_i} \right| \to 0$$

Path Continuity: Since Brownian motion paths are almost surely continuous.

$$\therefore \quad \frac{Q_{\Pi_n}}{\max_{1 \le i < n} \left(\left| W_{t_{i+1}} - W_{t_i} \right| \right)} \le TV_{\Pi_n}$$

$$\therefore \text{ as } \|\Pi_n\| \to 0 \qquad TV_{\Pi_n} \ge \infty$$

Key Takeaways

- Riemann integral is built from first-order increments $\Delta t = t_i - t_{i-1}$ and works for smooth / finite-variation paths.
- Riemann–Stieltjes integral $\int f\,dg$ typically requires the integrator g to have bounded variation.
- For Brownian motion, we obtain

$$Q_{\Pi_n} \xrightarrow{L^2} T, \qquad Q_{\Pi_n} \xrightarrow{a.s} T, \qquad TV_{\Pi_n} \to \infty.$$

- Hence Brownian paths are not finite-variation paths, so treating W_t with ordinary Riemann / Riemann–Stieltjes calculus is not sufficient.

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भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Towards $\int_0^T f(t)\, dW_t$

From Riemann–Stieltjes integral

$$\int_0^T f(t)\, dg(t) \text{ is defined for any continuous integrand } f$$

when the integrator g has bounded variation, i.e.,

$$\sum_i (\Delta g_i)^2 = \sum_i \big(g(t_{i+1}) - g(t_i)\big)^2 = 0$$

what if $g(t) = W_t$?
We are interested in

$$\int_0^T f(t)\, dW_t \quad - \quad \text{Weighted Random accumulation over time}$$

Why Riemann–Stieltjes Fails for Brownian Motion

but from quadratic variation of Brownian motion, We have

$$\sum_{i=0}^{n-1} (\Delta W_{t_i})^2 = \sum_{i=0}^{n-1} \left(W_{t_{i+1}} - W_{t_i}\right)^2 = T$$

$\therefore$ Riemann Stieltjes Integral doesn't exist.

Wiener Integral

Consider f is a deterministic function where $f \in L^2[0,T]$.
$L^2[0,T]$ denotes hilbert space of all real valued squared integrable functions on $[0,T]$.
Consider a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T$$

$\{W_t\}_{t\geq 0}$ is a brownian motion with respect to filtration

$$\mathscr{F}_t = \sigma(W_s : s \leq t)$$

Consider f is a simple function

$$f(t) = \sum_{i=0}^{n-1} a_i \, \mathbf{1}_{[t_i, t_{i+1})}(t) \qquad \forall i,\ a_i \in \mathbb{R}$$

Wiener Integral(cont.)

$$f(t) = \begin{cases} a_0, & t \in [t_0, t_1) \\ a_1, & t \in [t_1, t_2) \\ \vdots & \\ a_n, & t \in [t_n, t_{n+1}) \end{cases}$$

Define

$$I(f)(\omega) = \sum_{i=0}^{n-1} f(t_i)\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)$$

Wiener Integral(cont.)

$$I(f) = \sum_{i=0}^{n-1} f(t_i)\left(W_{t_{i+1}} - W_{t_i}\right)$$

$$I(f) = \sum_{i=0}^{n-1} a_i\left(W_{t_{i+1}} - W_{t_i}\right)$$

$I(f)$ is a random variable and measurable $\mathscr{F}_{t_n}$.

Expectation of Wiener Integral

$$\mathbb{E}[I(f)] = \mathbb{E}\left[\sum_{i=0}^{n-1} a_i \left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

$$= \sum_{i=0}^{n-1} \mathbb{E}\left[a_i \left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

$$= \sum_{i=0}^{n-1} a_i \, \mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

$$\therefore \boxed{\mathbb{E}[I(f)] = 0}$$

$\therefore$

Expectation of $(I(f))^2$

$$\mathbb{E}[(I(f))^2] = \mathbb{E}\left[\left(\sum_{i=0}^{n-1} a_i \left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_i a_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right]$$

for $i \neq j$

$$\mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right] = 0$$

for $i = j$

$$\mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right] = t_{i+1} - t_i$$

Expectation of $(I(f))^2$ (cont.)

$$\mathbb{E}[(I(f))^2] = \sum_{i=0}^{n-1} a_i^2 \, (t_{i+1} - t_i)$$
$$= \sum_{i=0}^{n-1} (f(t_i))^2 \, (t_{i+1} - t_i)$$

$$\boxed{\mathbb{E}[(I(f))^2] = \int_0^T (f(t))^2 \, dt}$$

Note: The limit is not required here because for a step function is already "built out of rectangles" So integration is just adding those rectangles.

Linearity of Wiener Integral

If $f(t)$ and $g(t)$ are deterministic functions.
where

$$f(t) = \sum_{i=0}^{n-1} a_i \, \mathbf{1}_{[t_i, t_{i+1})}, \qquad g(t) = \sum_{j=0}^{m-1} b_j \, \mathbf{1}_{[s_j, s_{j+1})}$$

$$0 = t_0 < t_1 < \cdots < t_n = T \qquad \text{and} \qquad 0 = s_0 < s_1 < \cdots < s_m = T$$

are the two partitions with respect to $f(t)$ and $g(t)$.

$$I(h) = \sum_{i=0}^{n-1} h_i \left(W_{t_{i+1}} - W_{t_i} \right) \qquad \forall \, i, \; h_i \in \mathbb{R}$$

Need to show

$$I(af(t) + bg(t)) = aI(f) + bI(g) \qquad \text{for } a, b \in \mathbb{R}$$

Linearity of Wiener Integral (cont.)

proof:

Given

$$I(f) = \sum_{i=0}^{n-1} a_i \left(W_{t_{i+1}} - W_{t_i}\right) \qquad I(g) = \sum_{j=0}^{m-1} b_j \left(W_{s_{j+1}} - W_{s_j}\right)$$

Consider a partition

$$\Pi = \{u_0, u_1, \ldots, u_K\} = \{t_0, t_1, \ldots, t_n\} \cup \{s_0, s_1, \ldots, s_m\}$$

Now, rewrite $f(t)$ and $g(t)$ w.r.t Π

$$f(t) = \sum_{k=0}^{K-1} \alpha_k \mathbf{1}_{[u_k, u_{k+1})} \qquad g(t) = \sum_{k=0}^{K-1} \beta_k \mathbf{1}_{[u_k, u_{k+1})}$$

$$I(f) = \sum_{k=0}^{K-1} \alpha_k \left(W_{u_{k+1}} - W_{u_k}\right) \qquad I(g) = \sum_{k=0}^{K-1} \beta_k \left(W_{u_{k+1}} - W_{u_k}\right)$$

Linearity of Wiener Integral (cont.)

Consider

$$W_{t_{i+1}} - W_{t_i} = \sum_{k:\,[u_k,u_{k+1})\subset[t_i,t_{i+1})} \left(W_{u_{k+1}} - W_{u_k}\right)$$

$$I(f) = \sum_{i=0}^{n-1} a_i \sum_{k:\,[u_k,u_{k+1})\subset[t_i,t_{i+1})} \left(W_{u_{k+1}} - W_{u_k}\right)$$

$$= \sum_{k=0}^{K-1} \alpha_k \left(W_{u_{k+1}} - W_{u_k}\right)$$

Similarly

$$I(g) = \sum_{k=0}^{K-1} \beta_k \left(W_{u_{k+1}} - W_{u_k}\right)$$

Linearity of Wiener Integral (cont.)

$$
\begin{aligned}
af + bg &= a \sum_{k=0}^{K-1} \alpha_k \, \mathbf{1}_{[u_k, u_{k+1})} + b \sum_{k=0}^{K-1} \beta_k \, \mathbf{1}_{[u_k, u_{k+1})} \\
&= \sum_{k=0}^{K-1} (a\alpha_k + b\beta_k) \, \mathbf{1}_{[u_k, u_{k+1})} \\
\therefore \; I(af + bg) &= \sum_{k=0}^{K-1} (a\alpha_k + b\beta_k) \left(W_{u_{k+1}} - W_{u_k} \right) \\
&= a \sum_{k=0}^{K-1} \alpha_k \left(W_{u_{k+1}} - W_{u_k} \right) + b \sum_{k=0}^{K-1} \beta_k \left(W_{u_{k+1}} - W_{u_k} \right) \\
&= a \, I(f) + b \, I(g)
\end{aligned}
$$

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$I(f)$ is well defined by showing L^2 convergence

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Indian Institute of Technology Hyderabad

Let $f \in L^2[0,T]$. Choose a sequence $\{f_n\}_{n=1}^{\infty}$ of step function such that $f_n \to f$ in $L^2[0,T]$. Need to show

$$I(f) = \lim_{n\to\infty} I(f_n) \qquad \text{in } L^2(\Omega)$$

Suppose $\{g_m\}$ is another such sequence i.e $g_m \to f$ in $L^2[0,T]$.

$$\therefore\ \mathbb{E}\left[(I(f_n) - I(g_m))^2\right] = \mathbb{E}\left[(I(f_n - g_m))^2\right] \qquad \text{(linearity)}$$

$$= \int_0^T (f_n(t) - g_m(t))^2\, dt \qquad \text{(isometry)}$$

$$f_n(t) - g_m(t) = f_n(t) - f(t) - (g_m(t) - f(t))$$

$$(f_n(t) - g_m(t))^2 \leq 2\left[(f_n(t) - f(t))^2 + (g_m(t) - f(t))^2\right]$$

Wiener integral via L^2 convergence

$$\therefore \int_0^T (f_n(t) - g_m(t))^2 \, dt \leq 2 \int_0^T \left[(f_n(t) - f(t))^2 + (g_m(t) - f(t))^2 \right] dt \to 0, \text{ as } n, m \to \infty$$

So, $\{I(f_n)\}_{n=1}^{\infty}$ is cauchy in $L^2(\Omega)$

$$\therefore I(f)(\omega) = \left(\int_0^T f(t) \, dW_t \right)(\omega) \qquad \omega \in \Omega$$

$$I = \int_0^T f(t) \, dW_t$$

where

$$I : L^2[0, T] \longrightarrow L^2(\Omega)$$

We use L^2 convergence to define the integral, defining almost surely for every outcome $\omega \in \Omega$ is difficult due to irregular nature of Brownian motion.

Wiener Integral: Key Interpretations

Wiener Integral:

$$I_t = \int_0^t f(s)\, dW_s$$

- **Financial Interpretation:**
  $f(t)$ = Number of Shares held at time t
  W_t = Price fluctuations
  Wiener Integral $\Rightarrow$ Total gain/loss from trading
- **No Drift (Mean):**

$$E[I(t)] = 0$$

$\Rightarrow$ No predictable trend

- **Variance:**

$$E[I(t)^2] = \int_0^t f(s)^2 ds$$

- **Distribution:**

$$I(t) \sim \mathcal{N}\left(0, \int_0^t f(s)^2 ds\right)$$

$\Rightarrow$ Gaussian (if f deterministic)

- **Special Case:** If $f(t) = 1$, then

$$I(t) = W_t$$

Ito Integral

Ito Integral:

Consider a probability space $(\Omega, \mathscr{F}, P)$.
W_t is a brownian motion adapted to filtration $\mathscr{F}_t$
where $t \in [0, T]$ and

$$\mathscr{F}_t = \sigma(W_s : 0 \le s \le t)$$

Consider a stochastic process X_t
where

$$X_t(\omega) \in L^2_{ad}([0, T] \times \Omega) \text{ i.e.,}$$

(i) $X_t(\omega)$ is adapted to brownian motion filtration $\mathscr{F}_t$

(ii) $\int_0^T \mathbb{E}\left[|X_t|^2\right] < \infty$

Ito integral (cont.)

Consider X_t is a simple process

$$X_t(\omega) = \sum_{i=0}^{n-1} \xi_i(\omega)\, \mathbf{1}_{[t_i, t_{i+1})}(t) \qquad \omega \in \Omega$$

$$X_t = \begin{cases} \xi_0, & t \in [t_0, t_1) \\ \xi_1, & t \in [t_1, t_2) \\ \vdots & \\ \xi_n, & t \in [t_n, t_{n+1}) \end{cases}$$

where ξ_i is $\mathscr{F}_{t_i}$ measurable
and $\mathbb{E}[\xi_i^2] < \infty$

Ito integral (cont.)

Define

$$I_t(\omega) = \sum_{i=0}^{n-1} \xi_i(\omega) \left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right) \qquad \omega \in \Omega$$

I_t is a random variable and $\mathscr{F}_t$ measurable.

Expectation of Ito Integral

$$\mathbb{E}[I_t] = \sum_{i=0}^{m-1} \mathbb{E}\left[\xi_i \left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

ξ_i is $\mathscr{F}_{t_i}$ measurable
$W_{t_{i+1}} - W_{t_i}$ is $\mathscr{F}_{t_{i+1}}$ measurable.

$\therefore$ ξ_i and $W_{t_{i+1}} - W_{t_i}$ are independent random variables

$$\therefore \ \mathbb{E}[I_t] = \sum_{i=0}^{m-1} \mathbb{E}[\xi_i] \cdot \mathbb{E}[W_{t_{i+1}} - W_{t_i}]$$

$$\mathbb{E}[I_t] = 0$$

Expectation of I_t^2

$$\mathbb{E}[I_t^2] = \mathbb{E}\left[\left(\sum_{i=0}^{m-1} \xi_i \left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \xi_i\xi_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right]$$

for $i \neq j$

$$\mathbb{E}\left[\xi_i\xi_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right]$$

ξ_i is $\mathscr{F}_{t_i}$ measurable and ξ_j is $\mathscr{F}_{t_j}$ measurable
$W_{t_{i+1}} - W_{t_i}$ is $\mathscr{F}_{t_{i+1}}$ measurable $W_{t_{j+1}} - W_{t_j}$ is $\mathscr{F}_{t_{j+1}}$ measurable
$\xi_i, \xi_j, W_{t_{i+1}} - W_{t_i}$ and $W_{t_{j+1}} - W_{t_j}$ are independent

Expectation of I_t^2 (cont.)

$$\therefore\ \mathbb{E}\left[\xi_i\xi_j\left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right] = 0$$

for $i = j$

$$\begin{aligned}\mathbb{E}\left[\xi_i^2\left(W_{t_{i+1}} - W_{t_i}\right)^2\right] &= \mathbb{E}\left[\xi_i^2\right]\mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right]\\ &= \mathbb{E}\left[\xi_i^2\right]\left(t_{i+1} - t_i\right)\\ &= \mathbb{E}\left[X_{t_i}^2\right]\left(t_{i+1} - t_i\right)\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}[I_t^2] &= \sum_{i=0}^{m-1}\mathbb{E}\left[X_{t_i}^2(t_{i+1} - t_i)\right]\\ &= \mathbb{E}\left[\sum_{i=0}^{m-1}X_{t_i}^2(t_{i+1} - t_i)\right] = \mathbb{E}\left[\int_0^t X_s^2\,ds\right]\end{aligned}$$

Linearity of Ito Integral

$$3.\ I_t(aX_t + bY_t) = a\, I_t(X_t) + b\, I_t(Y_t)$$

$$X_t = \sum_{i=0}^{n-1} \xi_i\, \mathbf{1}_{[t_i, t_{i+1})}(t)$$

$$Y_t = \sum_{j=0}^{m-1} \eta_j\, \mathbf{1}_{[s_j, s_{j+1})}(t)$$

From linearity of wiener integral, we have seen any 2 partitions of 2 different simple functions we can come up with common refinement

Linearity of Ito Integral (cont.)

$$\therefore \quad X_t = \sum_{k=0}^{K-1} \xi_k \, \mathbf{1}_{[u_k, u_{k+1})}(t) \qquad Y_t = \sum_{k=0}^{K-1} \eta_k \, \mathbf{1}_{[u_k, u_{k+1})}(t)$$

$$I_t(X_t) = \sum_{k=0}^{K-1} \xi_k \left(W_{u_{k+1}} - W_{u_k}\right) \qquad I_t(Y_t) = \sum_{k=0}^{K-1} \eta_k \left(W_{u_{k+1}} - W_{u_k}\right)$$

now

$$aX_t + bY_t = \sum_{k=0}^{K-1} (a\,\xi_k + b\,\eta_k)\, \mathbf{1}_{[u_k, u_{k+1})}$$

$$I_t(aX_t + bY_t) = \sum_{i=0}^{n-1} (a\,\xi_i + b\,\eta_i)\left(W_{t_{i+1}} - W_{t_i}\right)$$

Linearity of Ito integral (cont.)

$$
\begin{aligned}
&= a \sum_{i=0}^{n-1} \xi_i \left(W_{t_{i+1}} - W_{t_i}\right) + b \sum_{i=0}^{m-1} \eta_i \left(W_{t_{i+1}} - W_{t_i}\right) \\
&= a\, I_t(X_t) + b\, I_t(Y_t)
\end{aligned}
$$

i.e,

$$
\int_0^T (aX_t + bY_t)\, dW_t = a \int_0^T X_t\, dW_t + b \int_0^T Y_t\, dW_t
$$

Convergence and Existence in $L^2(\Omega)$

We define the Itô integral of the general process f as the limit of the integrals of the approximating step processes:

$$I(f) = \lim_{n\to\infty} I(f_n).$$

(For the above approximating step processes, one can refer to 4.3 in Introduction to Stochastic Integration by Hui-Hsiung-Kuo and 3.1 in Stochastic Differential Equations by Bernt Oksendal)

To prove this limit exists, we show $\{I(f_n)\}$ is a Cauchy sequence in $L^2(\Omega)$. By the Itô Isometry:

$$E\left[|I(f_n) - I(f_m)|^2\right] = \int_a^b E\left[|f_n(t) - f_m(t)|^2\right] dt.$$

As $n, m \to \infty$, the right-hand side goes to zero because $\{f_n\}$ is a Cauchy sequence in the space of integrands.

Because $L^2(\Omega)$ is a complete Hilbert space, every Cauchy sequence must converge to a unique limit that also belongs to $L^2(\Omega)$.

$$I_t = \int_0^t X_s \, dW_s$$

where

$$I_t : \; L^2_{ad}[T, \Omega] \longrightarrow L^2(\Omega)$$

Ito integral (Martingale)

I_t is a martingale:

i) I_t is adapted
ii) $\mathbb{E}[I_t] < \infty \; (= 0)$
iii) $\mathbb{E}[I_t \mid \mathscr{F}_s] = I_s$ for $s < t$
proof:

$$I_t = \int_0^t X_u \, dW_u = \sum_i X_i (W_{t_{i+1}} - W_{t_i})$$

for $s < t$, decompose

$$\begin{aligned} I_t &= \int_0^s X_u \, dW_u + \int_s^t X_u \, dW_u \\ &= I_s + \int_s^t X_u \, dW_u \end{aligned}$$

Ito integral (Martingale cont.)

now

$$
\begin{aligned}
\mathbb{E}[I_t \mid \mathscr{F}_s] &= \mathbb{E}\left[I_s + \int_s^t X_u \, dW_u \mid \mathscr{F}_s\right] \\
&= \mathbb{E}[I_s \mid \mathscr{F}_s] + \mathbb{E}\left[\int_s^t X_u \, dW_u \mid \mathscr{F}_s\right] \\
&= I_s + \mathbb{E}\left[\int_s^t X_u \, dW_u \mid \mathscr{F}_s\right] to(1)
\end{aligned}
$$

Ito integral (Martingale cont.)

we know

$$X_u = \sum_{i=0}^{m-1} X_i \, \mathbf{1}_{[t_i, t_{i+1})} \qquad \text{and } X_i \in \mathscr{F}_{t_i}$$

assume $s = t_k$ and $t = t_m$ such that $t_k \leq t_m$

$$\int_s^t X_u \, dW_u = \sum_{i=k}^{m-1} X_i (W_{t_{i+1}} - W_{t_i})$$

$$\mathbb{E}\left[\int_s^t X_u \, dW_u \mid \mathscr{F}_s\right] = \mathbb{E}\left[\sum_{i=k}^{m-1} X_i (W_{t_{i+1}} - W_{t_i}) \mid \mathscr{F}_s\right]$$

Ito integral (Martingale cont.)

for $i = k \;\Rightarrow$

$$\begin{aligned}\mathbb{E}\left[X_k(W_{t_{k+1}} - W_{t_k}) \mid \mathscr{F}_s\right] &= X_k\,\mathbb{E}\left[(W_{t_{k+1}} - W_{t_k}) \mid \mathscr{F}_{t_k}\right] \\ &= 0 \;\rightarrow (2)\end{aligned}$$

for $i > k$

$$\begin{aligned}\mathbb{E}\left[X_i(W_{t_{i+1}} - W_{t_i}) \mid \mathscr{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[X_i(W_{t_{i+1}} - W_{t_i}) \mid \mathscr{F}_{t_i}\right] \mid \mathscr{F}_s\right] \\ &= \mathbb{E}\left[X_i\,\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i}) \mid \mathscr{F}_{t_i}\right] \mid \mathscr{F}_s\right] \\ &= \mathbb{E}\left[X_i\left(\mathbb{E}\left[W_{t_{i+1}} \mid \mathscr{F}_{t_i}\right]\mathbb{E}\left[W_{t_i} \mid \mathscr{F}_{t_i}\right]\right) \mid \mathscr{F}_s\right] \\ &= 0 \;\rightarrow (3)\end{aligned}$$

from $(1), (2), (3)$

$$\boxed{\mathbb{E}[I_t \mid \mathscr{F}_s] = I_s}$$

Quadratic variation of Ito integral

Consider a partition Π_n in $[0, t]$ where $0 = t_0 < t_1 < \cdots < t_n = t$.
Consider sample quadratic variation of Ito integral as

$$
\begin{aligned}
I_{\Pi_n} &= \sum_{i=0}^{n-1} (I_{t_{i+1}} - I_{t_i})^2 \\
&= \sum_{i=0}^{n-1} \left(\int_0^{t_{i+1}} X_s \, dW_s - \int_0^{t_i} X_s \, dW_s \right)^2 \\
&= \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} X_s \, dW_s \right)^2
\end{aligned}
$$

Quadratic variation of Ito integral (cont.)

From the definition of Ito integrable

$$I_t = \int_0^t X_s \, dW_s = \lim_{\|\Pi_n\| \to 0} \sum X_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right)$$

from $\|\Pi_n\| \to 0$, we can say that each interval becomes arbitrarily small.
Now consider a partition in the sub-interval $[t_i, t_{i+1}]$ where $X_s = X_{t_i}$.

$$\Pi_m = \{s_0, s_1, \ldots, s_m\} \qquad s_0 = t_i, \ldots, s_m = t_{i+1}$$

$$I_{\Pi_n} = \lim_{\|\Pi_m\| \to 0} I_{\Pi_m}^{(t_0)} + \lim_{\|\Pi_m\| \to 0} I_{\Pi_m}^{(t_1)} + \cdots + \lim_{\|\Pi_m\| \to 0} I_{\Pi_m}^{(t_{n-1})}$$

$$I_{\Pi_n} = \sum_{i=0}^{n-1} \lim_{\|\Pi_m\| \to 0} I_{\Pi_m}^{(t_i)}$$

Quadratic variation of Ito integral (cont.)

$$I_{\Pi_m}^{(t_i)} = \sum_{j=0}^{m-1} \left(X_{t_i}\left(W_{s_{j+1}} - W_{s_j}\right)\right)^2 \qquad \because \text{ Evalation at left end point}$$

$$= X_{t_i}^2 \sum_{j=0}^{m-1} (W_{s_{j+1}} - W_{s_j})^2$$

$$= X_{t_i}^2 \left(t_{i+1} - t_i\right) \qquad \because \; s_0 = t_i, \ldots, s_m = t_{i+1}, \qquad Q_{\Pi_n} \xrightarrow{a.s} T$$

$$I_{\Pi_n} = \sum_{i=0}^{n-1} X_{t_i}^2 \left(t_{i+1} - t_i\right)$$

Quadratic variation of Ito integral (cont.)

$$
\begin{aligned}
\lim_{\|\Pi_n\| \to 0} I_{\Pi_n} &= \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} X_{t_i}^2 \left(t_{i+1} - t_i\right) \\
&= \int_0^t X_s^2 \, ds \qquad \text{(from Riemann integral)}
\end{aligned}
$$

Ito process

In any real world processes, the processes has both drift(Predictable) and randomness(Unpredictable).

Ito Process

Consider a probability space $(\Omega, \mathscr{F}, P)$.
W_t is a brownian motion.
X_t is said to be Ito process

$$X_t = X_0 + \int_0^t m_s\, ds + \int_0^t \sigma_s\, dW_s$$

where m_t and σ_t are adapted wrt filtration of Brownian motion.

Ito process (conditions)

For the Ito process

$$X_t = X_0 + \int_0^t m_s \, ds + \int_0^t \sigma_s \, dW_s,$$

Ito process to be well defined the below conditions need to be satisfy:

1. $\int_0^t |m_t| \, dt < \infty$ almost surely for every $t > 0$.
2. $\int_0^t \mathbb{E}[|\sigma_t|^2] \, dt < \infty$ for every $t > 0$.
3. X_0 is $\mathscr{F}_0$ measurable, and $\int_0^t m_s \, ds$, $\int_0^t \sigma_s \, dW_s$ are $\mathscr{F}_t$ measurable.

Hence X_t is $\mathscr{F}_t$ measurable.

Ito process (cont.)

Consider the increment over a small interval

$$X_{t+\Delta t} - X_t = (X_0 - X_0) + \int_t^{t+\Delta t} m_s\, ds + \int_t^{t+\Delta t} \sigma_s\, dW_s$$
$$= m_t\, \Delta t + \sigma_t\, \Delta W_t$$

$$dX_t = m_t\, dt + \sigma_t\, dW_t$$

This is differential form of Ito process Note: The differential form of informal notation because
dW$_t hasnomathematicalmeaninganditactsasstochasticintegratorintheintegralform.$

Ito lemma/Ito Doeblin formula

Ito lemma:

For a function $f(t,x)$, The taylor expansion is given as

$$f(t+dt,x+dx) = f(t,x) + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \cdots$$

Consider a function X_t which has non zero quadratic variation function

$$f(t+dt,X_t+dX_t) = f(t,X_t) + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

For brownian motion

$$(dW_t)^2 = dt \qquad (dt)^2 = 0$$

$$(dt)(dW_t) = (dt)^{3/2} = 0, \qquad (dt)^{3/2} \ll dt$$

Ito lemma (cont.)

Consider

$$\therefore \boxed{df(t, X_t) = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dt} \rightarrow (1)$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} dt - \text{Ito correction term}$$

It appears because Brownian motion has non-zero quadratic variation,

$$(dW_t)^2 = dt,$$

so second-order terms do not vanish as they do in ordinary calculus.

Ito lemma (cont.)

from Ito process

$$dX_t = m_t\, dt + \sigma_t\, dW_t \ \rightarrow (2)$$

$$\begin{aligned}(dX_t)^2 &= (m_t dt + \sigma_t dW_t)^2 \\ &= m_t^2 (dt)^2 + \sigma_t^2 (dW_t)^2 + 2 m_t \sigma_t (dW_t)(dt) \\ &= \sigma_t^2\, dt \ \rightarrow (3)\end{aligned}$$

Substitute $(2), (3)$ in (1)

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}(m_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dt$$

$$df = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 + \frac{\partial f}{\partial x} m_t) dt + \frac{\partial f}{\partial x}\sigma_t dW_t$$

Ito lemma (cont.)

1. The function f need to be C^2.
2. If f is C^2 function then $f(X_t)$ is also a Ito process, where X_t is a Ito process.
3. Ito lemma after applying for any $f(X_t)$ where X_t is an ito process and f is C^2 then $f(t, X_t)$ is also Ito process.

$$df = \underbrace{\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 + \frac{\partial f}{\partial x} m_t\right)}_{\text{drift}} dt + \underbrace{\frac{\partial f}{\partial x}\sigma_t}_{\text{volatility}} dW_t$$

4. dW_t is also an ito process with drift 0 and volatility 1.

Examples of using Ito Lemma

Examples:

1. $f(X_t) = X_t^2$

$$d(X_t^2) = 2X_t m_t dt + 2X_t \sigma_t dW_t + \sigma_t^2 dt$$

2. $f(X_t) = e^{X_t}$

$$de^{X_t} = e^{X_t}\left(m_t + \tfrac{1}{2}\sigma_t^2\right)dt + e^{X_t}\sigma_t dW_t$$

3. $f(X_t) = \ln X_t$

$$d(\ln X_t) = \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2$$

Examples of using Ito Lemma (cont.)

4. For Brownian motion W_t, take $f(t,x) = x^2$.

$$
\begin{aligned}
d(W_t^2) &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x}\, dW_t \\
&= \left(0 + \frac{1}{2}\cdot 2\right) dt + 2W_t\, dW_t \\
&= dt + 2W_t\, dW_t
\end{aligned}
$$

Hence

$$2W_t\, dW_t = d(W_t^2) - dt$$

$$\boxed{\int_0^t W_s\, dW_s = \frac{1}{2}(W_t^2 - t)}$$

Geometric Brownian Motion

Geometric Brownian Motion:

Consider an differential form of ito process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$\mu = \text{drift (growth rate)}$$

$$\sigma = \text{volatility}$$

let $f(S_t) = \ln S_t$
from Ito lemma

$$df(X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2$$

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Geometric Brownian Motion (lemma applied)

|| नवोन्मेषेण देशनिर्माणम् ||
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$$df(S_t) = 0 + \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2$$

$$(dS_t)^2 = \sigma^2 S_t^2 (dW_t)^2 = \sigma^2 S_t^2 dt$$

Geometric Brownian Motion (cont.)

$$\begin{aligned} \therefore\ df(S_t) &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \end{aligned}$$

Geometric Brownian Motion (cont.)

$$\therefore \; d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

Integrating 0 to t

$$\ln S_t - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)$$

$$\ln S_t \sim \mathcal{N}\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right)$$

Geometric Brownian Motion (distribution)

$$
\begin{aligned}
\because \quad & W_t \sim \mathcal{N}(0, t), \\
\therefore \quad & \sigma W_t \sim \mathcal{N}(0, \sigma^2 t), \\
\therefore \quad & a + \sigma W_t \sim \mathcal{N}(a, \sigma^2 t), \\
\therefore \quad & S_t \text{ is log-normal.}
\end{aligned}
$$

Reason behind Geometric Brownian motion

Reason behind Geometric Brownian motion

If S_t is Stock price at given time t

1. Brownian motion as stock price can go negative but Geometric Brownian Motion (GBM) gives the solution always positive
2. To talk about percentage return instead of raw prices