Absolute Continuity of measures

Definition

Consider a measurable space $(\Omega, \mathcal{F})$.
Let $\mu : \mathcal{F} \to [0, +\infty]$ and $\nu : \mathcal{F} \to [0, +\infty]$ be two measures.

We say ν is absolutely continuous with respect to μ if

$$\text{if } \mu(A) = 0, \text{ then } \nu(A) = 0 \qquad \forall A \in \mathcal{F}$$

Radon-Nikodym Theorem

Definition

Consider a measurable space $(\Omega, \mathcal{F})$.
Let $\mu : \mathcal{F} \to [0, +\infty]$ and $\nu : \mathcal{F} \to [0, +\infty]$ be two measures.
Suppose that $\nu \ll \mu$.
Then, there exists a non-negative, measurable function $f : \Omega \to [0, +\infty]$.

$$\nu(A) = \int_A f \, d\mu = \int_\Omega 1_A f \, d\mu, \qquad \forall A \in \mathcal{F}$$

Probability-measure form

Let's consider a measurable space $(\Omega, \mathcal{F})$.
$\mathbb{P}$ and $\mathbb{Q}$ are 2 probability measures on the space $(\Omega, \mathcal{F})$.
Suppose that $\mathbb{Q} \ll \mathbb{P}$ then

$$\mathbb{Q}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}[1_A Z]$$

where Z is a non negative random variable.
For a finite sample space,

$$Z(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}$$

This is just a density of one measure wrt other.

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Radon Nikodym Derivative process

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Definition

Consider a measurable space $(\Omega, \mathcal{F})$.
$\mathbb{P}$ and $\mathbb{Q}$ are 2 Probability measures.
Consider filtration $\mathcal{F}_t$ i.e, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$.

Radon Nikodym derivative process given as

$$M_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}$$

(RND of $\mathbb{Q}$ wrt $\mathbb{P}$ restricted to $\mathcal{F}_t$)
(density of $\mathbb{Q}$ relative to $\mathbb{P}$ wrt information upto time t)

M_t is Radon Nikodym derivative process which is a stochastic process.

Radon Nikodym Derivative process (cont.)

Properties:

1. Measurability:

$$M_t \text{ is } \mathcal{F}_t \text{ measurable}$$

2. $M_t \geq 0$ as density of one measure wrt another is non negative.
3. $\mathbb{E}_{\mathbb{P}}[M_t] = 1$

From Radon-Nikodym,

$$\mathbb{Q}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega)$$

at $A = \Omega$,

$$\mathbb{Q}(\Omega) = \int_\Omega M_t(\omega)\, d\mathbb{P}(\omega)$$

$$1 = \mathbb{E}_{\mathbb{P}}[M_t]$$

Martingale property

4. Martingale property:

$$\mathbb{E}_{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right] = M_s \qquad (s < t)$$

For any $A \in \mathcal{F}_s$,

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[1_A M_t] = \mathbb{E}_{\mathbb{P}}[1_A M_s] \to (1)$$

Now consider

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[1_A M_t] &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[1_A M_t \mid \mathcal{F}_s\right]\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[1_A\, \mathbb{E}_{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right]\right], \qquad \forall A \in \mathcal{F}_s \end{aligned}$$

$$\therefore\ \mathbb{E}_{\mathbb{P}}\left[1_A\, \mathbb{E}_{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right]\right] = \mathbb{E}_{\mathbb{P}}[1_A M_s], \qquad \text{from (1)}$$

$$\therefore\ M_s = \mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s]$$

Existence and Uniquness of measure $\mathbb{Q}$

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
M_t is a martingale with respect to filtration $\mathcal{F}_t$ and also consider $M_t \geq 0$ and $\mathbb{E}_{\mathbb{P}}[M_t] = 1$ then there exist a probability measure $\mathbb{Q}$ such that

$$M_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$$

proof:
Define

$$\mathbb{Q}(A) := \mathbb{E}_{\mathbb{P}}[1_A M_t] \qquad A \in \mathcal{F}_t$$

1. $\mathbb{Q}$ is a probability measure:

$$M_t \geq 0 \text{ and } 1_A \geq 0 \implies \mathbb{E}_{\mathbb{P}}[1_A M_t] \geq 0$$

(i) $\mathbb{Q}(\varnothing) = \mathbb{E}_{\mathbb{P}}[1_{\varnothing} M_t] = 0$
(ii) $\mathbb{Q}(\Omega) = \mathbb{E}_{\mathbb{P}}[1_{\Omega} M_t] = \mathbb{E}_{\mathbb{P}}[M_t] = 1$

Existence and Uniquness of measure $\mathbb{Q}$ (cont.)

(iii) Let $A_1, A_2, \ldots$ are countable disjoint set

$$\mathbb{Q}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{\infty} 1_{A_i} M_t\right]$$

$$= \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}}\left[1_{A_i} M_t\right]$$

$$= \sum_{i=1}^{\infty} \mathbb{Q}(A_i)$$

$\therefore$ $\mathbb{Q}$ is a probability measure

Existence and Uniquness of measure $\mathbb{Q}$ (cont.)

2. Absolute continuity:

If $\mathbb{P}(A) = 0$ then $1_A = 0$ a.s.

$$\therefore\ \mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[0 \cdot M_t] = 0 \text{ a.s.}$$

$$\therefore\ \text{If } \mathbb{P}(A) = 0 \text{ then } \mathbb{Q}(A) = 0$$

$$\therefore\ \mathbb{Q} \ll \mathbb{P}$$

From Radon Nikodym Theorem

$$\therefore\ \mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[1_A M_t] \qquad A \in \mathcal{F}_t, \qquad \therefore\ \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = M_t$$

$$\therefore\ \text{Existence is proved}$$

Existence and Uniquness of measure $\mathbb{Q}$ (cont.)

Suppose $\mathbb{Q}_1, \mathbb{Q}_2$ are 2 measures satisfy

$$\left.\frac{d\mathbb{Q}_1}{d\mathbb{P}}\right|_{\mathcal{F}_t} = M_t \qquad \text{and} \qquad \left.\frac{d\mathbb{Q}_2}{d\mathbb{P}}\right|_{\mathcal{F}_t} = M_t$$

then for any $A \in \mathcal{F}_t$

$$\mathbb{Q}_1(A) = \mathbb{E}_{\mathbb{P}}[1_A M_t]$$

$$= \mathbb{Q}_2(A)$$

$$\therefore\ \mathbb{Q}_1(A) = \mathbb{Q}_2(A) \qquad A \in \mathcal{F}_t$$

$$\therefore\ \mathbb{Q} \text{ is unique}$$

Expectation under measure $\mathbb{Q}$

For any random variable Y the expected value of Y with respect to measure $\mathbb{Q}$ is given as

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[ZY]$$

where Z is non-negative random variable.

Proof: For any measurable event A, the probability under $\mathbb{Q}$ by Radon-Nikodym Theorem given as

$$\mathbb{Q}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}[Z\, 1_A]$$

Proof of Case 1

Case 1: For a finite sample space

$$Z(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}$$

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[Y] &= \sum_{\omega \in \Omega} Y(\omega)\mathbb{Q}(\omega) \\
&= \sum_{\omega \in \Omega} Y(\omega)Z(\omega)\mathbb{P}(\omega) \\
&= \sum_{\omega \in \Omega} (Y(\omega)Z(\omega))\mathbb{P}(\omega) \\
&= \mathbb{E}_{\mathbb{P}}[YZ]
\end{aligned}$$

Proof of Case 2

Case 2: For any general uncountable probability spaces

(i) Indicator Function: If $Y = 1_A$

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{Q}}[1_A] = \mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[1_A Z]$$

(ii) Simple functions:

$$Y = \sum_{i=1}^{n} a_i \, 1_{A_i}, \qquad \forall i \in \mathbb{N},\ A_i \in \mathcal{F},\ a_i \in \mathbb{R}$$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[Y] &= \sum_{i=1}^{n} a_i \, \mathbb{E}_{\mathbb{Q}}[1_{A_i}] \\ &= \sum_{i=1}^{n} a_i \, \mathbb{E}_{\mathbb{P}}[Z \, 1_{A_i}] = \mathbb{E}_{\mathbb{P}}\left[Z \sum_{i=1}^{n} a_i 1_{A_i}\right] = \mathbb{E}_{\mathbb{P}}[ZY] \end{aligned}$$

Proof of Case 2(cont.)

(iii) Non negative random variable:
Consider Y is a non negative random variable Y.
For any non-negative Borel-measurable function Y can be considered as the limit of an increasing sequence of simple function $\{Y_n\}$
i.e, $0 \leq Y_1 \leq Y_2 \leq \cdots \leq Y$ and $\lim_{n\to\infty} Y_n(\omega) = Y(\omega)$.

$$\mathbb{E}_{\mathbb{Q}}[Y_n] = \mathbb{E}_{\mathbb{P}}[ZY_n]$$

Proof of Case 2(cont.)

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}[Y_n] = \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[ZY_n]$$

By monotone convergence theorem

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[ZY]$$

(iv) General Integrable Function:

For any general Integrable function Y can be written as

$$Y = Y^+ - Y^-$$

where

$$Y^+ = \max\{Y, 0\}, \qquad Y^- = \max\{-Y, 0\}$$

and Y^+ and Y^- are non negative random variables.

Proof of Case 2(cont.)

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[Y] &= \mathbb{E}_{\mathbb{Q}}[Y^+ - Y^-] \\
&= \mathbb{E}_{\mathbb{Q}}[Y^+] - \mathbb{E}_{\mathbb{Q}}[Y^-] \\
&= \mathbb{E}_{\mathbb{P}}[ZY^+] - \mathbb{E}_{\mathbb{P}}[ZY^-] \\
&= \mathbb{E}_{\mathbb{P}}[Z(Y^+ - Y^-)] \\
&= \mathbb{E}_{\mathbb{P}}[ZY] \qquad \text{if } \mathbb{E}[|ZY|] < \infty
\end{aligned}
$$

$$
\therefore \boxed{\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[ZY]}
$$

Conditional expectation under measure $\mathbb{Q}$

Definition

Let $s < t$ and let Y be an $\mathcal{F}_t$ measurable random variable and M_t is Radon Nikodym Derivative process. Then

$$\mathbb{E}_{\mathbb{Q}}\left[Y \mid \mathcal{F}_s\right] = \frac{1}{M_s}\mathbb{E}_{\mathbb{P}}\left[YM_t \mid \mathcal{F}_s\right] \qquad \text{for } M_s > 0$$

Proof: Let

$$Z = \frac{1}{M_s}\mathbb{E}_{\mathbb{P}}\left[YM_t \mid \mathcal{F}_s\right]$$

We must show two things:

1. Z is $\mathcal{F}_s$ measurable.
2. For every $A \in \mathcal{F}_s$,

$$\mathbb{E}_{\mathbb{Q}}[1_A Y] = \mathbb{E}_{\mathbb{Q}}[1_A Z]$$

Conditional expectation under measure $\mathbb{Q}$ (cont.)

(i) Measurability:

1. M_s is $\mathcal{F}_s$ measurable by definition of the RND process.
2. $\mathbb{E}_{\mathbb{P}}[YM_t \mid \mathcal{F}_s]$ is $\mathcal{F}_s$ measurable by definition of conditional expectation.

$$\therefore \; Z \text{ is } \mathcal{F}_s \text{ measurable}$$

(ii) Consider an event $A \in \mathcal{F}_s$.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[1_A Y] &= \mathbb{E}_{\mathbb{P}}[YM_t 1_A] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}[YM_t 1_A \mid \mathcal{F}_s]\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[1_A \, \mathbb{E}_{\mathbb{P}}[YM_t \mid \mathcal{F}_s]\Big] \rightarrow (1)
\end{aligned}$$

Conditional expectation under measure $\mathbb{Q}$ (cont.)

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[Z 1_A] &= \mathbb{E}_{\mathbb{Q}}\left[1_A \frac{1}{M_s} \mathbb{E}_{\mathbb{P}}[Y M_t \mid \mathcal{F}_s]\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[1_A M_s \frac{1}{M_s} \mathbb{E}_{\mathbb{P}}[Y M_t \mid \mathcal{F}_s]\right] \\
&= \mathbb{E}_{\mathbb{P}}\Big[1_A \, \mathbb{E}_{\mathbb{P}}[Y M_t \mid \mathcal{F}_s]\Big] \rightarrow (2)
\end{aligned}$$

From (1) and (2),

$$\mathbb{E}_{\mathbb{Q}}[1_A Y] = \mathbb{E}_{\mathbb{Q}}[Z 1_A]$$

$$\therefore \boxed{\mathbb{E}_{\mathbb{Q}}[Y \mid \mathcal{F}_s] = \frac{1}{M_s} \mathbb{E}_{\mathbb{P}}[Y M_t \mid \mathcal{F}_s]}$$

Exponential Martingale

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
M_t and W_t are continuous time martingale and Brownian motion respectively

Definition 1

Define

$$Z_t := \exp\left(uM_t - \frac{1}{2}[M_t]\right)$$

where $[M_t]$ is the quadratic variation of continuous time martingale.
Z_t is called exponential martingale.

Exponential Martingale (cont.)

Definition 2

Consider θ_s is an adapted process wrt $\mathcal{F}_t$

$$Z_t := \exp\left(\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right)$$

Z_t is a local martingale.

For a true martingale, it should satisfy Novikov's condition
i.e, For a given time t

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \theta_s^2 \, ds\right)\right] < \infty$$

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Exponential Martingale

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Now consider $\theta_s = u \in \mathbb{R}$ a constant scalar

$$\therefore\ Z_t := \exp\left(uW_t - \frac{1}{2}u^2 t\right)$$

This Z_t is equivalent to definition 1, when $M_t = W_t$

Definition 3

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
$\{W_t\}_{t\geq 0}$ is a Brownian motion.
Z_t is an exponential martingale defined as

$$Z_t = \exp\left(uW_t - \frac{1}{2}u^2 t\right)$$

where Z_t is a martingale and $u \in \mathbb{R}$

Exponential Martingale (cont.)

Martingale proof:

$$
\begin{aligned}
Z_t &= \exp\left(u(W_t - W_s) + uW_s - \frac{1}{2}u^2 t\right) \\
&= \exp\big(u(W_t - W_s)\big) \cdot \exp\left(uW_s - \frac{1}{2}u^2 t\right) \\
\mathbb{E}[Z_t \mid \mathcal{F}_s] &= \mathbb{E}\left[\exp\big(u(W_t - W_s)\big) \cdot \exp\left(uW_s - \frac{1}{2}u^2 t\right) \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\exp\big(u(W_t - W_s)\big)\right] \cdot \exp\left(uW_s - \frac{1}{2}u^2 t\right) \\
\mathbb{E}[e^{uX}] &= \exp\left(\frac{1}{2}u^2\sigma^2\right), \qquad X \sim N(0, \sigma^2) \quad \text{by MGF}
\end{aligned}
$$

Exponential Martingale (cont.)

$$
\begin{aligned}
\therefore \quad \mathbb{E}\left[Z_t \mid \mathcal{F}_s\right] &= \exp\left(\frac{1}{2}u^2(t-s)\right) \cdot \exp\left(uW_s - \frac{1}{2}u^2 t\right) \\
&= \exp\left(uW_s - \frac{1}{2}u^2 s\right) \\
&= Z_s
\end{aligned}
$$

Exponential Martingale (cont.)

Applying Ito lemma:

$$f(t, W_t) = \exp\left(uW_t - \frac{1}{2}u^2 t\right) = Z_t$$

$$\begin{aligned} df(t, W_t) &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial W_t}(dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}(dW_t)^2 \\ &= Z_t\left(-\frac{1}{2}u^2\right) dt + Z_t u\, dW_t + \frac{1}{2} Z_t u^2\, dt \end{aligned}$$

$$\boxed{dZ_t = uZ_t\, dW_t} \qquad \text{(martingale)}$$

Note: The term $-\frac{1}{2}u^2 t$ in Z_t is a “drift correction term”.

Levy Characterization of Brownian Motion

Theorem

A stochastic process M_t is a Brownian motion if and only if there exist a probability measure $\mathbb{Q}$ and a filtration $\mathcal{F}_t$ such that

- M_t is a continuous martingale with respect to $\mathcal{F}_t$ under $\mathbb{Q}$.
- $\mathbb{Q}(M_0 = 0) = 1$.
- The quadratic variation satisfies $[M]_t = t \quad \forall t \geq 0$.

Levy Characterization of Brownian Motion (cont.)

Proof:

The necessary part follows from standard Brownian motion,

- M_t is a martingale.
- The process start at zero almost surely.
- The quadratic variation is $[M]_t = t$.

For sufficiency, use the exponential martingale

$$Z_t = \exp\left(uM_t - \frac{1}{2}u^2 t\right) \qquad \text{with } \mathbb{Q}(M_0 = 0) = 1$$

Consider $u = i\lambda$,

$$Z_t = \exp\left(i\lambda M_t + \frac{1}{2}\lambda^2 t\right)$$

Levy Characterization of Brownian Motion (cont.)

$$\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = M_s$$

$$\mathbb{E}\left[\exp\left(i\lambda M_t + \frac{1}{2}\lambda^2 t\right)\middle| \mathcal{F}_s\right] = \exp\left(i\lambda M_s + \frac{1}{2}\lambda^2 s\right)$$

$$\exp\left(i\lambda M_s + \frac{1}{2}\lambda^2 t\right)\mathbb{E}\left[\exp\left(i\lambda(M_t - M_s)\right)\middle| \mathcal{F}_s\right] = \exp\left(i\lambda M_s + \frac{1}{2}\lambda^2 s\right)$$

$$\therefore \quad \mathbb{E}\left[\exp\left(i\lambda(M_t - M_s)\right)\middle| \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}\lambda^2(t-s)\right)$$

Levy Characterization of Brownian Motion (cont.)

The conditional characteristic function is given by

$$\mathbb{E}\left[e^{i\lambda X} \mid \mathcal{F}\right] = e^{-\frac{1}{2}\lambda^2\sigma^2}, \qquad X \sim N(0, \sigma^2)$$

and X is independent of $\mathcal{F}$.

$$\therefore \quad M_t - M_s \sim N(0, t - s)$$

Girsanov's Theorem

Theorem

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
$\{W_t\}_{t\geq 0}$ be a standard Brownian motion
Define exponential martingale Z_t satisfies novikov's condition

$$Z_t = \exp\left(\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right)$$

Define a probability measure

$$\mathbb{Q}(A) = \int_A Z_t \, d\mathbb{P} \qquad \forall A \in \mathcal{F}_t$$

Girsanov's Theorem (cont.)

Theorem (continued)

The theorem states that under probability measure $\mathbb{Q}$, the process defined by

$$\widetilde{W}_t = W_t - \int_0^t \theta_s \, ds$$

is a standard Brownian motion.

Girsanov's Theorem Proof

From Levy's characterization of Brownian Motion, to show $\widetilde{W}_t$ is a Brownian motion, need to prove following three

1. $\mathbb{Q}(\widetilde{W}_0 = 0) = 1.$
2. Quadratic variation is t, for any $\widetilde{W}_t$, $t \geq 0$.
3. $\widetilde{W}_t$ is a continuous martingale with the respect to $\mathcal{F}_t$ under $\mathbb{Q}$.

Girsanov's Theorem Proof (cont.)

1.
$$\widetilde{W}(0) = W(0) - \int_0^0 \theta_s \, ds = 0$$

2.
$$\widetilde{W}_t = W_t - \int_0^t \theta_s \, ds$$

The differential form of the above is

$$d\widetilde{W}_t = dW_t - \theta_t \, dt$$

Girsanov's Theorem Proof (cont.)

$$
\begin{aligned}
(d\widetilde{W}_t)^2 &= (dW_t)^2 - 2\theta_t\, dW_t\, dt + \theta_t^2 (dt)^2 \\
&= dt
\end{aligned}
$$

$$
\therefore \quad \widetilde{W}_t \text{ has a quadratic variation } dt
$$

3. Need to show

$$
\mathbb{E}_{\mathbb{Q}}\left[\widetilde{W}_t \mid \mathcal{F}_s\right] = \widetilde{W}_s
$$

We know that,

$$
\mathbb{E}_{\mathbb{Q}}\left[\widetilde{W}_t \mid \mathcal{F}_s\right] = \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}\left[\widetilde{W}_t Z_t \mid \mathcal{F}_s\right]
$$

Girsanov's Theorem Proof (cont.)

Consider $X_t = \widetilde{W}_t Z_t$.

$$dX_t = d(\widetilde{W}_t Z_t)$$

By Ito product rule

$$dX_t = Z_t\, d\widetilde{W}_t + \widetilde{W}_t\, dZ_t + d\widetilde{W}_t\, dZ_t \qquad \rightarrow (1)$$

Girsanov's Theorem Proof (cont.)

and also we know that

$$dZ_t = \theta_t Z_t \, dW_t \qquad \rightarrow (2)$$

$$d\widetilde{W}_t = dW_t - \theta_t \, dt \qquad \rightarrow (3)$$

Substitute (2) and (3) in (1)

$$\begin{aligned} dX_t &= Z_t \, dW_t - Z_t\theta_t \, dt + \theta_t Z_t \widetilde{W}_t \, dW_t + \theta_t Z_t (dW_t)^2 - \theta_t^2 Z_t (dW_t)(dt) \\ &= (1 + \theta_t \widetilde{W}_t) Z_t \, dW_t \end{aligned}$$

Girsanov's Theorem Proof (cont.)

From Ito above equation, we can say dX_t is a stochastic integral.
W_t is a Brownian motion under probability measure $\mathbb{P}$.

$$\therefore \quad X_t \text{ is a martingale i.e., } \widetilde{W}_t Z_t \text{ is a } \mathbb{P}\text{-martingale}$$

$$\therefore \quad \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}\left[\widetilde{W}_t Z_t \mid \mathcal{F}_s\right] = \frac{1}{Z_s}(\widetilde{W}_s Z_s) = \widetilde{W}_s$$

$$\therefore \quad \mathbb{E}_{\mathbb{Q}}\left[\widetilde{W}_t \mid \mathcal{F}_s\right] = \widetilde{W}_s$$

Girsanov's Theorem Proof (cont.)

$\therefore$ By Levy's characterization of Brownian motion,

$\widetilde{W}_t$ is a standard Brownian motion under probability measure $\mathbb{Q}$.

Arbitrage

Arbitrage: There exist a way to make a guaranteed profit with no initial capital.

Fundamental Theorem of Asset Pricing

A market model is free of arbitrage if and only if there exists an Equivalent Martingale Measure.

Violation of No arbitrage:

1. Violation of the Binomial No arbitrage Condition

Let the value of the stock at time $t = 0$ is S_0. At time $t = 1$, it can move up by a factor u and move down by a factor d.

Fundamental Theorem of Asset Pricing (cont.)

$$\begin{tikzcd}[row sep=small] & S_1 = uS_0 \\ S_0 \arrow[ur] \arrow[dr] & \\ & S_1 = dS_0 \end{tikzcd}$$

Consider a risk free rate r, at time $t = 1$, the value of stock with respect to risk free rate is $S_0 + rS_0$ or $S_0(1 + r)$.
To prevent arbitrage, the below condition should hold

$$\boxed{0 < d < 1 + r < u}$$

Violation of No Arbitrage

What if $d \geq 1 + r$:

Consider an investor can construct a portfolio starting with zero initial wealth.

1. At $t = 0$, borrow an amount equal to stock price (S_0) from the money market with interest rate r.
2. Purchase stock (S_0).
3. At $t = 1$,

Case (i): stock goes down by factor d

$$S_1 = dS_0$$

$$\begin{aligned} \text{profit} &= dS_0 - (1 + r)S_0 \\ &= (d - (1 + r))S_0 \geq 0 \qquad \because\ d \geq 1 + r \end{aligned}$$

Violation of No Arbitrage (cont.)

Case (ii): Stock goes up by factor u

$$S_1 = uS_0$$

$$\text{profit} = uS_0 - (1+r)S_0 > 0 \qquad \because\ d < u \text{ and } d \geq 1+r$$

Violation of No Arbitrage (cont.)

What if $u \leq 1 + r$:

1. At $t = 0$, sell exactly one share of the stock that do not own at the current price S_0. This is called Short the stock.
2. Immediately invest S_0 in the risk free rate (r).
3. At time $t = 1$, the market you invested has a value $S_0(1 + r)$.
4. At $t = 1$,

Case (i):

$$S_1 = uS_0$$

$$\text{profit}: \quad S_0(1+r) - uS_0 \geq 0 \qquad (uS_0 \text{ to cover short})$$

Case (ii):

$$S_1 = dS_0$$

$$\text{profit}: \quad S_0(1+r) - dS_0 > 0 \qquad (dS_0 \text{ to cover short})$$

Motivation for Change of Measure

In practice, investors demand extra returns for holding risky asset.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A risky asset S_t follows geometric Brownian motion under measure $\mathbb{P}$.

$$\text{i.e.,} \quad dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Motivation for Change of Measure (cont.)

μ is average rate of growth of stock

σ is volatility

In general,

$$\mu = r + \text{risk premium}$$

where r is risk free interest rate.

And also $\mu > r$ and risk premium depends on investor's risk.
For derivative pricing we do not want to price to depend on investor's risk preference.

Motivation for Change of Measure (cont.)

So, we go for risk neutral measure $(\mathbb{Q})$ and risk premium with respect to investor choice is adjusted and considered stock grows at risk free rate 'r'.

(This doesn't mean investors stop wanting extra return. It means risk premium is absorbed into change of measure.)

Discounting

Discounting:

If an amount A grows to a future value through interest, discounting takes the future value and brings it back to time zero to see what it is worth in today's term. The resulting value is called the "discounted value".

$$\frac{dA(t)}{dt} = rA(t)$$

$$\frac{1}{A(t)}\,dA(t) = r\,dt$$

$$\log A(t) = rt + c$$

Discounting (cont.)

At $t = 0$,

$$\log A(0) = c, \qquad e^c = A(0)$$

$$A(t) = A(0)e^{rt}$$
$$A(0) = A(t)e^{-rt}$$

The current discounted price is $A(t)e^{-rt}$.
This tells present worth of a future amount A at time t is $A(t)e^{-rt}$.

Discounting (cont.)

By the Fundamental Theorem of Asset Pricing, in an arbitrage-free market there exists an equivalent martingale measure $\mathbb{Q}$ under which every discounted traded asset is a martingale

Consider s is present instance, t is future instance.

$$\text{i.e.,} \quad \mathbb{E}_{\mathbb{Q}}\left[e^{-rt}S_t \mid \mathcal{F}_s\right] = e^{-rs}S_s \qquad \rightarrow (1)$$

Equivalently,

$$S_s = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(t-s)}S_t \mid \mathcal{F}_s\right]$$

Let

$$\widetilde{S}_t = e^{-rt}S_t$$

From (1)

$$\mathbb{E}\left[\widetilde{S}_t \mid \mathcal{F}_s\right] = \widetilde{S}_s$$

Discounting (cont.)

$\widetilde{S}_t$ must be a martingale under risk neutral measure $\mathbb{Q}$.

Discounting enables

1. To move from real measure $\mathbb{P}$ to risk neutral measure $\mathbb{Q}$.
2. Discounting makes martingale under measure $\mathbb{Q}$.
3. Risk neutral measure (or EMM) under this the market is free of arbitrage.

Risk Neutral Measure

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Consider a risky asset S_t follows a geometric brownian motion under measure $\mathbb{P}$

$$\therefore \quad dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

μ is drift and σ is volatility.

Consider Risk free asset (B_t) which may be a bond or money market account that grows at the constant interest rate r.

$$dB_t = rB_t \, dt \quad \Longrightarrow \quad B_t = e^{rt} \quad \text{where } B_0 = 1.$$

Risk Neutral Measure (cont.)

Consider a discounted process (D_t) to account for the time value of money, we define the discount factor as the reciprocal of the bond price.

$$D_t = \frac{1}{B_t} = e^{-rt}$$

$$\text{i.e.,} \quad dD_t = -rD_t \, dt$$

Risk Neutral Measure (cont.)

Under measure $\mathbb{Q}$,

$$\mathbb{E}_{\mathbb{Q}}\left[\widetilde{S}_t \mid \mathcal{F}_s\right] = \widetilde{S}_s$$

Now,
we know that

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Consider

$$f(t, S_t) = e^{-rt} S_t = \widetilde{S}_t$$

we know Ito lemma

$$df(t, S_t) = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial x}(dS_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dS_t)^2$$

Risk Neutral Measure (cont.)

$$\frac{\partial f}{\partial t} = -re^{-rt}S_t = -r\widetilde{S}_t, \qquad \frac{\partial f}{\partial x} = e^{-rt}, \qquad \frac{\partial^2 f}{\partial x^2} = 0$$

$$\begin{aligned} d(e^{-rt}S_t) &= -r\widetilde{S}_t\,dt + e^{-rt}(\mu S_t\,dt + \sigma S_t\,dW_t) \\ &= -r\widetilde{S}_t\,dt + \mu\widetilde{S}_t\,dt + \sigma\widetilde{S}_t\,dW_t \\ &= (\mu - r)\widetilde{S}_t\,dt + \sigma\widetilde{S}_t(d\widetilde{W}_t + \theta_t\,dt) \end{aligned}$$

$$d\widetilde{S}_t = (\mu - r + \sigma\theta_t)\widetilde{S}_t\,dt + \sigma\widetilde{S}_t\,d\widetilde{W}_t$$

Risk Neutral Measure (cont.)

we know that $\widetilde{S}_t$ is a martingale

$$\therefore \quad \mu - r + \sigma\theta_t = 0$$

$$\boxed{\theta_t = \frac{r-\mu}{\sigma}}$$

Risk Neutral Measure (cont.)

Now consider

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

$$\begin{aligned} dS_t &= \mu S_t \, dt + \sigma S_t (d\widetilde{W}_t + \theta_t \, dt) \\ &= \mu S_t \, dt + \sigma S_t \, d\widetilde{W}_t + \sigma S_t \left(\frac{r - \mu}{\sigma} \right) dt \end{aligned}$$

$$\boxed{dS_t = r S_t \, dt + \sigma S_t \, d\widetilde{W}_t}$$

$\therefore$ under probability measure $\mathbb{Q}$, the stock grows at risk free rate r.

Risk Neutral Measure (cont.)

Consider,

$$d\widetilde{S}_t = d(D_t S_t)$$

from Ito product rule

$$d\widetilde{S}_t = S_t\, dD_t + D_t\, dS_t + (dD_t)(dS_t)$$

Substitute dS_t and dD_t

$$\begin{aligned} d\widetilde{S}_t &= D_t(rS_t\, dt + \sigma S_t\, d\widetilde{W}_t) + S_t(-rD_t\, dt) + 0 \\ &= \sigma S_t D_t\, d\widetilde{W}_t \\ &= \sigma \widetilde{S}_t\, d\widetilde{W}_t \qquad \because S_t D_t = \widetilde{S}_t \end{aligned}$$

$\therefore \quad \widetilde{S}_t$ is the discounted stock price and is a martingale under $\mathbb{Q}$.

Martingale Representation Theorem (MRT)

Theorem

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
$\{W_t\}_{0\le t\le T}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ is the filtration w.r.t. $\{W_t\}$.

If $\{M_t\}_{0\le t\le T}$ is a square integrable martingale

(i.e., $\mathbb{E}[M_t^2] < \infty$)

with respect to $\mathcal{F}_t$, then there exist a unique process such that

$$M_t = M_0 + \int_0^t \Gamma_s \, dW_s$$